NOTE ON GRADED IDEALS WITH LINEAR FREE RESOLUTION AND LINEAR QUOTIENS IN THE EXTERIOR ALGEBRA

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Abstract: The goal of this note is to study graded ideals with linear free resolution and linear quotients in the exterior algebra. We use an extension of the notion of linear quotients, namely componentwise linear quotients, to give another proof of the well-known result that an ideal with linear quotients is componentwise linear. After that, we consider special cases where a product of linear ideals has a linear free resolution.

1 Introduction

Let K be a field and V an n-dimensional K-vector space, where $n \geq 1$, with a fixed basis $e_1, \ldots, e_n$. We denote by $E = K\langle e_1, \ldots, e_n \rangle$ the exterior algebra of V. It is a standard graded K-algebra with defining relations $v \wedge v = 0$ for all $v \in V$ and graded components $E_i = \Lambda^i V$ by setting $\deg e_i = 1$. Let M be a finitely generated graded left and right E-module satisfying the equations

$$um = (-1)^{\deg u \deg m} mu$$

for homogeneous elements $u \in E$, $m \in M$. The category of such E-modules M is denoted by $\mathcal{M}$. For a module $M \in \mathcal{M}$, the minimal graded free resolution of M is uniquely determined and it is an exact sequence of the form

$$\ldots \longrightarrow \bigoplus_{j \in \mathbb{Z}} E(-j)^{\beta^E_{1,j}(M)} \longrightarrow \bigoplus_{j \in \mathbb{Z}} E(-j)^{\beta^E_{0,j}(M)} \longrightarrow M \longrightarrow 0.$$

Note that $\beta^E_{i,j}(M) = \dim_K \operatorname{Tor}^E_i(K, M)_j$ for all $i, j \in \mathbb{Z}$. We call the numbers $\beta^E_{i,j}(M)$ the *graded Betti numbers* of M. The module M is said to have a *d-linear resolution* if $\beta^E_{i,i+j}(M) = 0$ for all i and $j \neq d$. This is equivalent to the condition that M is generated in degree d and all non-zero entries in the matrices representing the differential maps are of degree one. Following [5], M is called *componentwise linear* if the submodules $M_{\langle i \rangle}$ of M generated by M_i has an i-linear resolution for all $i \in \mathbb{Z}$. Furthermore, M is said to have *linear quotients* with respect to a homogeneous system of generators $m_1, \ldots, m_r$ if $(m_1, \ldots, m_{i-1}) :_E m_i$ is a linear ideal, i.e., an ideal in E generated by linear forms, for $i = 1, \ldots, r$. We say that M has *componentwise linear quotients* if each submodule $M_{\langle i \rangle}$ of M has linear quotients w.r.t. some of its minimal systems of homogeneous generators for all $i \in \mathbb{Z}$ such that $M_i \neq 0$.

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This paper is devoted to the study of the structure of a minimal graded free resolution of graded ideals in E. More precisely, we are interested in graded ideals which have d-linear resolutions, linear quotients or are componentwise linear. It is well-known that a graded ideal that has linear quotients w.r.t. a minimal system of generators is componentwise linear (see [10; Corollary 2.4] for the polynomial ring case and [9; Theorem 5.4.5] for the exterior algebra case). We give another proof for this result in Corollary 3.5 by using Theorem 3.4 which states that if a graded ideal has linear quotients then it has componentwise linear quotients.

Motivated by a result of Conca and Herzog in [3; Theorem 3.1] that a product of linear ideals in a polynomial ring has a linear resolution, we study in Section 4 the problem whether this result holds or not in the exterior algebra. At first, we get a positive answer for the case the linear ideals are generated by variables (Theorem 4.2). Then we consider some other special cases (Proposition 4.5, 4.6) when this result also holds.

2 Preliminaries

We present in this section some homological properties of graded modules in $\mathcal{M}$ related to resolutions and componentwise linear property.

Let $M \in \mathcal{M}$. The (Castelnuovo-Mumford) *regularity* for a graded module $M \in \mathcal{M}$ is given by

$$\operatorname{reg}_E(M) = \max\{j - i : \beta^E_{i,j}(M) \neq 0\} \text{ for } M \neq 0 \text{ and } \operatorname{reg}_E(0) = -\infty.$$

For every $0 \neq M \in \mathcal{M}$, one can show that $t(M) \leq \operatorname{reg}_E(M) \leq d(M)$ (see [9; Section 2.1]). So $\operatorname{reg}_E(M)$ is always a finite number for every $M \neq 0$.

Note that for a graded ideal $J \neq 0$, by the above definitions one has $\operatorname{reg}_E(E/J) = \operatorname{reg}_E(J) - 1$. This can be seen indeed by the fact that if $F_\bullet \longrightarrow J$ is the minimal graded free resolution of J, then $F_\bullet \longrightarrow E \longrightarrow E/J$ is the minimal graded free resolution of E/J.

For a short exact sequence $0 \to M \to N \to P \to 0$ of non-zero modules in $\mathcal{M}$, there are relationships among the regularities of its modules by evaluating in Tor-modules in the long exact sequence

$$\begin{aligned} \ldots \longrightarrow \operatorname{Tor}^E_{i+1}(P,K)_{i+1+j-1} \longrightarrow \ & \operatorname{Tor}^E_i(M,K)_{i+j} \longrightarrow \operatorname{Tor}^E_i(N,K)_{i+j} \longrightarrow \\ & \operatorname{Tor}^E_i(P,K)_{i+j} \longrightarrow \operatorname{Tor}^E_{i-1}(M,K)_{i-1+j+1} \longrightarrow \ldots \end{aligned}$$

More precisely, one has:

Lemma 2.1. *Let* $0 \to M \to N \to P \to 0$ *be a short exact sequence of non-zero modules in* $\mathcal{M}$. *Then:*

(i) $\operatorname{reg}_E(N) \leq \max\{\operatorname{reg}_E(M), \operatorname{reg}_E(P)\}$.

(ii) $\operatorname{reg}_E(M) \leq \max\{\operatorname{reg}_E(N), \operatorname{reg}_E(P) + 1\}$.

(iii) $\operatorname{reg}_E(P) \leq \max\{\operatorname{reg}_E(N), \operatorname{reg}_E(M) - 1\}$.

Next we recall some facts about componentwise linear ideals and linear quotients in the exterior algebra. Componentwise linearity was defined for ideals over the polynomial ring by Herzog and Hibi in [6] to characterize a class of simplicial complexes, namely, sequentially Cohen-Macaulay simplicial complexes. Such ideals have been received a lot of attention in several articles, e.g., [2], [4], [8], [10]. All materials in this section can be found in the book by Herzog and Hibi (see [5; Chapter 8]) or Kämpf's dissertation (see [9; Section 5.3, 5.4]).

Definition 2.2. Let $M \in \mathcal{M}$ be a finitely generated graded E-module. Recall that M has a d-linear resolution if $\beta^E_{i,i+j}(M) = 0$ for all i and all $j \neq d$. Following [5] we call M *componentwise linear* if the submodules $M_{\langle i \rangle}$ of M generated by M_i has an i-linear resolution for all $i \in \mathbb{Z}$.

Note that a componentwise linear module which is generated in one degree has a linear resolution. A module that has a linear resolution is componentwise linear.

At first, for an ideal with a linear resolution one has the following property.

Lemma 2.3 ([9; Lemma 5.3.4]). *Let $0 \neq J \subset E$ be a graded ideal. If J has a d-linear resolution, then $\mathfrak{m}J$ has a $(d+1)$-linear resolution.*

Next we recall some facts about ideals with linear quotients over the exterior algebra. For more details, one can see [9; Section 5.4].

Definition 2.4. Let $J \subset E$ be a graded ideal with a system of homogeneous generators $G(J) = \{u_1, \dots, u_r\}$. Then J is said to have *linear quotients* with respect to $G(J)$ if $(u_1, \dots, u_{i-1}) :_E u_i$ is an ideal generated by linear forms for $i = 1, \dots, r$. We say that J has linear quotients if there exists a minimal system of homogeneous generators $G(J)$ such that J has linear quotients w.r.t. $G(J)$.

Note that for the definition of linear quotients over the exterior algebra, we need the condition that $0 :_E u_1$ has to be generated by linear forms, i.e., u_1 is a product of linear forms. This condition is omitted in the definition of linear quotients over the polynomial ring.

Remark 2.5. Let J be a graded ideal with linear quotients w.r.t. $G(J) = \{u_1, \dots, u_r\}$. Then $\deg(u_i) \geq \min\{\deg(u_1), \dots, \deg(u_{i-1})\}$. Indeed, assume the contrary that $\deg(u_i) < \min\{\deg(u_1), \dots, \deg(u_{i-1})\}$. Then there is a nonzero K-linear combination of u_j, $j = 1, \dots, i-1$, belonging to (u_i) since $(u_1, \dots, u_{i-1}) :_E u_i$ is generated by linear forms. Hence, we can omit one u_k in $\{u_1, \dots, u_{i-1}\}$ to get a smaller system of generators, this is a contradiction since $G(J)$ is a minimal.

3 Graded ideals with linear quotients

The goal of this section is to prove by another way the result that graded ideals with linear quotients are componentwise linear. For this, we use a so-called notion of componentwise linear quotients which is defined for monomial ideals over the polynomial ring by Jahan and Zheng in [8]. We also review matroidal ideals over an exterior algebra as important examples of ideals with linear quotients.

Let $J \subset E$ be a graded ideal with linear quotients and $u_1, \dots, u_r$ an admissible order of $G(J)$. Following [8], the order $u_1, \dots, u_r$ of $G(J)$ is called a *degree increasing admissible order* if $\deg u_i \leq \deg u_{i+1}$ for $i = 1, \dots, r$. By using exterior algebra's technics, we have the following lemmas which are similar to the ones for monomial ideals over the polynomial ring proved in [8] (note that we prove here for graded ideals).

Lemma 3.1. *Let $J \subset E$ be a graded ideal with linear quotients. Then there is a degree increasing admissible order of $G(J)$.*

Proof. We prove the statement by induction on r, the number of generators of J. It is clear for the case $r = 1$.

Assume $r > 1$ and $u_1, \dots, u_r$ is an admissible order. So $J = (u_1, \dots, u_{r-1})$ has linear quotients with the given order. By the induction hypothesis, we can assume that $\deg u_1 \leq \dots \leq \deg u_{r-1}$. We only need to consider the case $\deg u_r < \deg u_{r-1}$. Let i be the smallest integer such that $\deg u_{i+1} > \deg u_r$. It is clear that $i+1 \neq 1$ since $\deg u_1 = \min\{\deg u_1, \dots, \deg u_r\}$ by Remark 2.5. We now claim that $u_1, \dots, u_i, u_r, u_{i+1}, \dots, u_{r-1}$ is a degree increasing admissible order of $G(J)$. Indeed, we only need to prove that

$$(u_1, \dots, u_i) : u_r \text{ and } (u_1, \dots, u_i, u_r, u_{i+1}, \dots, u_{j-1}) : u_j$$

are generated by linear forms, for $j = i+1, \dots, r-1$.

At first, we claim that $(u_1, \dots, u_i) : u_r = (u_1, \dots, u_{r-1}) : u_r$ which is generated by linear forms since J has linear quotients w.r.t. $G(J)$. The inclusion $\subseteq$ is clear. Now let f be a linear form in $(u_1, \dots, u_{r-1}) : u_r$. Then $fu_r \in (u_1, \dots, u_{r-1})$. We get

$$fu_r = g + h, \text{ where } g \in (u_1, \dots, u_i) \text{ and } h \in (u_{i+1}, \dots, u_{r-1}).$$

Let $\deg u_r = d$. Then $\deg fu_r = d+1$ and $\deg u_j \geq d+1$ for $j = i+1, \dots, r-1$. So we can assume that $h \neq 0$ and $\deg g = \deg h = d+1$. This implies that h is a linear combination of some of $u_{i+1}, \dots, u_{r-1}$ and $h = fu_r - g \in (u_1, \dots, u_i, u_r)$. This contradicts the fact that $G(J)$ is a minimal system of generators. Hence $h = 0$ and we get $fu_r = g \in (u_1, \dots, u_i)$. Then $f \in (u_1, \dots, u_i) : u_r$. So $(u_1, \dots, u_i) : u_r = (u_1, \dots, u_{r-1}) : u_r$ is generated by linear forms.

Next let $i+1 \leq j \leq r-1$, we aim to show that

$$(u_1, \dots, u_i, u_r, u_{i+1}, \dots, u_{j-1}) : u_j = (u_1, \dots, u_i, u_{i+1}, \dots, u_{j-1}) : u_j$$

which is generated by linear forms. The inclusion $\supseteq$ is clear.

Let $f \in (u_1, \dots, u_i, u_r, u_{i+1}, \dots, u_{j-1}) : u_j$. We have

$$fu_j = g + hu_r, \text{ where } g \in (u_1, \dots, u_i, u_{i+1}, \dots, u_{j-1}) \text{ and } h \in E.$$

Then $fu_j - g = hu_r$. Therefore, $hu_r \in (u_1, \dots, u_i, u_{i+1}, \dots, u_{j-1}, u_j)$ and then

$$h \in (u_1, \dots, u_i, u_{i+1}, \dots, u_{j-1}, u_j) : u_r = (u_1, \dots, u_i) : u_r$$

by the above claim. Hence $hu_r \in (u_1, \dots, u_i)$ and $fu_j \in (u_1, \dots, u_i, u_{i+1}, \dots, u_{j-1})$. This implies $f \in (u_1, \dots, u_i, u_{i+1}, \dots, u_{j-1}) : u_j$ and we can conclude the proof.

Similar to Lemma 2.3, for ideals with linear quotients we have:

Lemma 3.2. *Let $J \subset E$ be a graded ideal. If J has linear quotients, then $\mathfrak{m}J$ has linear quotients.*

Proof. Let $G(J) = \{u_1, \dots, u_r\}$ be a minimal system of generators of J such that J has linear quotients w.r.t. $G(J)$. We prove the assertion by induction on r.

If $r = 1$, it is clear that the assertion holds. Now let $r > 1$, consider the set

$$B = \{u_1e_1, \dots, u_1e_n, u_2e_1, \dots, u_2e_n, \dots, u_re_1, \dots, u_re_n\}.$$

Then B is a system of generators of $\mathfrak{m}J$. Note that B is usually not the minimal system of generators. We claim that one can chose a subset of B which is a minimal system of generators of $\mathfrak{m}J$ and $\mathfrak{m}J$ has linear quotients w.r.t. this subset.

For $1 \le p \le r, 1 \le q \le n$, denote

$$J_{p,q} = \mathfrak{m}(u_1, \dots, u_{p-1}) + (u_pe_1, \dots, u_pe_{q-1}),$$

$$I_{p,q} = (u_1, \dots, u_{p-1}) : u_p + (e_1, \dots, e_q).$$

Note that $I_{p,q}$ is generated by linear forms. If $u_pe_q \in J_{p,q}$, then we remove u_pe_q from B. By this way, we get the minimal set

$$B' = \{u_ie_j : i = 1, \dots, r, j \in F_i\}.$$

Now we shall order B' in the following way: $u_{i_1}e_{j_1}$ comes before $u_{i_2}e_{j_2}$ if $i_1 < i_2$ or $i_1 = i_2$ and $j_1 < j_2$. By induction hypothesis, we have that $\mathfrak{m}(u_1, \dots, u_{r-1})$ has linear quotients w.r.t. the following system of generators

$$B'' = \{u_ie_j : i = 1, \dots, r-1, j \in F_i\} \subset B'.$$

Next let $j \in F_r$, it remains to show that $J_{r,j} : u_re_j$ is generated by linear forms. Indeed, we claim that $J_{r,j} : u_re_j = I_{r,j}$. Let $f = g + h \in I_{r,j}$, where $h \in (e_1, \dots, e_j)$ and $g \in (u_1, \dots, u_{r-1}) : u_r$. Then $h(u_re_j) \in (u_re_1, \dots, u_re_{j-1}) \subseteq J_{r,j}$. In addition,

$$g(u_re_j) = \pm e_j(gu_r) \in \mathfrak{m}(u_1, \dots, u_{r-1}) \subseteq J_{r,j}.$$

So we get $I_{r,j} \subseteq J_{r,j} : u_re_j$.

Now let $f \in J_{r,j} : u_re_j$, then $f(u_re_j) \in J_{r,j}$. Therefore, $fe_j \in J_{r,j} : u_r$. To ensure that $f \in I_{r,j}$ we only need to prove that:

(i) $J_{r,j} : u_r \subseteq I_{r,j-1}$,

(ii) $I_{r,j-1} : e_j = I_{r,j}$, i.e., e_j is a regular element on $I_{r,j-1}$.

To prove (i), let $g \in J_{r,j} : u_r$, then $gu_r \in J_{r,j}$. Hence $gu_r = h_1 + h_2u_r$, where $h_1 \in (u_1, \ldots, u_{r-1})$ and $h_2 \in (e_1, \ldots, e_{j-1})$. This implies that $(g - h_2)u_r \in (u_1, \ldots, u_{r-1})$. Thus $g - h_2 \in (u_1, \ldots, u_{r-1}) : u_r$. So we get $g \in I_{r,j-1}$ since $h_2 \in (e_1, \ldots, e_{j-1})$. Therefore, $J_{r,j} : u_r \subseteq I_{r,j-1}$.

To prove (ii), we note that $e_j \notin I_{r,j-1}$. Indeed, if $e_j \in I_{r,j-1}$, then

$$e_ju_r \in (u_1, \ldots, u_{r-1}) + (e_1, \ldots, e_{j-1})u_r.$$

It follows that

$$e_ju_r \in \mathfrak{m}(u_1, \ldots, u_{r-1}) + (e_1, \ldots, e_{j-1})u_r = J_{r,j}$$

since $\deg e_ju_r \geq \deg u_i + 1$ for $i = 1, \ldots, r-1$. This contradicts the fact that $e_ju_r \notin J_{r,j}$ because of the choice of B'. Therefore, e_j is a regular element on $I_{r,j-1}$ because of the fact that $I_{r,j-1}$ is a linear ideal and $e_j \notin I_{r,j-1}$.

Remark 3.3. Observe the following:

(i) The converse of the above lemma is not true. For instance, let $J = (e_{12}, e_{34}) \subset K\langle e_1, e_2, e_3, e_4\rangle$. Then $\mathfrak{m}J = (e_{123}, e_{124}, e_{134}, e_{234})$ has linear quotients in the given order, but J does not have linear quotients.

(ii) We cannot replace $\mathfrak{m}$ in the above lemma by a subset of variables. So we see that the product of two graded ideals with linear quotients need not have linear quotients again. For example, let $J = (e_{123}, e_{134}, e_{125}, e_{256})$ be a graded ideal in $K\langle e_1, \ldots, e_6\rangle$. Then we can check that J has linear quotients but $P = (e_1, e_2)J = (e_{1234}, e_{1256})$ has no linear quotients since P is generated in one degree and it does not have a linear resolution.

Recall that a graded ideal $J \subset E$ has *componentwise linear quotients* if each component of J has linear quotients. Now we are ready to prove the main result of this section.

Theorem 3.4. *Let $J \subset E$ be a graded ideal. If J has linear quotients, then J has componentwise linear quotients.*

Proof. By Lemma 3.1 and Lemma 3.2, we can assume that J is generated in two degrees d and $d+1$ and $G(J) = \{u_1, \ldots, u_p, v_1, \ldots, v_q\}$ is a minimal system of generators of J, where $\deg u_i = d$ for $i = 1, \ldots, p$ and $\deg v_j = d+1$ for $j = 1, \ldots, q$. By Lemma 3.1, we can also assume that $u_1, \ldots, u_p, v_1, \ldots, v_q$ is an admissible order, so $J_{\langle d\rangle}$ has linear quotients and then a linear resolution. We only need to prove that $J_{\langle d+1\rangle}$ has also linear quotients.

We have $J_{\langle d+1\rangle} = \mathfrak{m}(u_1, \ldots, u_p) + (v_1, \ldots, v_q)$. So we can assume that

$$G(J_{\langle d+1\rangle}) = \{w_1, \ldots, w_s, v_1, \ldots, v_q\},$$

where $w_1, \ldots, w_s$ is ordered as in Lemma 3.2 and the order is admissible. We only need to ensure that $(w_1, \ldots, w_s, v_1, \ldots, v_{i-1}) : v_i$ is generated by linear forms for $1 \leq i \leq q$. Indeed, we claim that

$$(w_1, \ldots, w_s, v_1, \ldots, v_{i-1}) : v_i = (u_1, \ldots, u_p, v_1, \ldots, v_{i-1}) : v_i, \tag{1}$$

which is generated by linear forms since J has linear quotients w.r.t. $G(J)$.

The inclusion "$\subseteq$" is clear. Now let $f \in (u_1, \dots, u_p, v_1, \dots, v_{i-1}) : v_i$, we have $fv_i \in (u_1, \dots, u_p, v_1, \dots, v_{i-1})$. So $fv_i = g+h$, where $g \in (u_1, \dots, u_p)$ and $h \in (v_1, \dots, v_{i-1})$. Since $\deg fv_i \geq d+1$, we can assume that $\deg g \geq d+1$. Moreover, $\deg u_j = d$ for $j = 1, \dots, p$, therefore $g \in \mathfrak{m}(u_1, \dots, u_p) = (w_1, \dots, w_s)$. Hence

$$fv_i \in (w_1, \dots, w_s, v_1, \dots, v_{i-1}) \text{ and then } f \in (w_1, \dots, w_s, v_1, \dots, v_{i-1}) : v_i.$$

This concludes the proof.

We get a direct consequence of this theorem which is analogous to a result over the polynomial ring of Sharifan and Varbaro in [10; Corollary 2.4]:

Corollary 3.5. *If $J \subset E$ is a graded ideal with linear quotients, then J is componentwise linear.*

The converse of Theorem 3.4 is still not known. However, we can prove the following:

Proposition 3.6. *Let $J \subset E$ be a graded ideal with componentwise linear quotients. Suppose that for each component $J_{\langle i \rangle}$ there exists an admissible order δ_i of $G(J_{\langle i \rangle})$ such that the elements of $G(\mathfrak{m}J_{\langle i-1 \rangle})$ form the initial part of δ_i. Then J has linear quotients.*

Proof. By the same argument as in the proof of Theorem 3.4, in particular, using the equation (1), we can confirm that J has linear quotients.

To conclude this section, we present a class of ideal with linear quotients, which will be used in the next section.

Example 3.7. A monomial ideal $J \subset E$ is said to be *matroidal* if it is generated in one degree and if it satisfies the following exchange property:

for all $u, v \in G(J)$, and all i with $i \in \operatorname{supp}(u) \setminus \operatorname{supp}(v)$, there exists an integer j with $j \in \operatorname{supp}(v) \setminus \operatorname{supp}(u)$ such that $(u/e_i)e_j \in G(J)$.

Now it is the same to the polynomial rings case that matroidal ideals have linear quotients. So a matroidal ideal is a componentwise linear ideal generated in one degree, hence it has a linear resolution. For the convenience of the reader we reproduce from [3; Proposition 5.2] the proof of this property. *Proof.* Let $J \subset E$ be a matroidal ideal. We aim to prove that J has linear quotients with respect to the reverse lexicographical order of the generators.

Let $u \in G(J)$ and let J_u be the ideal generated by all $v \in G(J)$ with $v > u$ in the reverse lexicographical order. Then we get

$$J_u : u = (v/[v, u] : v \in J_u) + \operatorname{ann}(u).$$

We claim that $J_u : u$ is generated by linear forms. Note that $\operatorname{ann}(u)$ is generated by linear forms which are variables appearing in u. So we only need to show that for each $v \in G(J)$ and $v > u$, there exists a variable $e_j \in J_u : u$ such that e_j divides $v/[v, u]$.

Let $u = e_1^{a_1} \dots e_n^{a_n}$ and $v = e_1^{b_1} \dots e_n^{b_n}$, where $0 \leq a_i, b_j \leq 1$ and $\deg u = \deg v$. Since $v > u$, there exist an integer i such that $a_i > b_i$ and $a_k = b_k$ for $k = i+1, \dots, n$. Moreover, J is a matroidal ideal and $i \in \operatorname{supp}(u) \setminus \operatorname{supp}(v)$, hence there exists an integer j such that $b_j > a_j$, or in other words, $j \in \operatorname{supp}(v) \setminus \operatorname{supp}(u)$, such that $u' = e_j(u/e_i) \in G(J)$. Then $ue_j = u'e_i$. Since $j < i$, we get $u' > u$ and $u' \in J_u$. Hence $e_j \in J_u : u$. Next by $j \in \operatorname{supp}(v) \setminus \operatorname{supp}(u) = \operatorname{supp}(v/[v,u])$, we have that e_j divides $v/[v,u]$, this concludes the proof.

4 Product of ideals with a linear free resolution

Motivated by a result of Conca and Herzog in [3] that the product of linear ideals (ideals generated by linear forms) over the polynomial ring has a linear resolution, we study in this section the following related problem:

Question 4.1. Let $J_1, \ldots, J_d \subseteq E$ be linear ideals. Is it true that the product $J = J_1 \ldots J_d$ has a linear resolution?

At first, by modifying the technic of Conca and Herzog in [3] for the exterior algebra, we get a positive answer to the above question for the case J_i is generated by variables for $i = 1, \ldots, d$.

Theorem 4.2. *The product of linear ideals which are generated by variables has a linear free resolution.*

Proof. Let $J_1, \ldots, J_d \subseteq E$ be ideals generated by variables and $J = J_1 \ldots J_d$. If $J = 0$, then the statement is trivial. We prove the statement for $J \neq 0$ by two ways. One uses properties of matroidal ideals and the other is a more conceptual proof.

Recall that a monomial ideal J is matroidal if it is generated in one degree such that for all $u, v \in G(J)$, and all i with $i \in \operatorname{supp}(u) \setminus \operatorname{supp}(v)$, there exists an integer j with $j \in \operatorname{supp}(v) \setminus \operatorname{supp}(u)$ such that $(u/e_i)e_j \in G(J)$. For the convenience of the reader, we present next the fact (following the proof of Conca and Herzog [3] in the polynomial ring case) that a product of two matroidal ideals over the exterior algebra is also a matroidal ideal. In fact, let I, J be matroidal ideals, $u, u_1 \in G(I)$ and $v, v_1 \in G(J)$ such that $uv, u_1v_1 \neq 0$ and $uv, u_1v_1 \in G(IJ)$. Let $i \in \operatorname{supp}(u_1v_1) \setminus \operatorname{supp}(uv)$. We need to show that there exists an integer $j \in \operatorname{supp}(uv) \setminus \operatorname{supp}(u_1v_1)$ with $(u_1v_1/e_i)e_j \in G(IJ)$.

Since $\operatorname{supp}(u_1v_1) = \operatorname{supp}(u_1) \cup \operatorname{supp}(v_1)$, without loss of generality, we may assume that $i \in \operatorname{supp}(u_1)$. Then $i \in \operatorname{supp}(u_1) \setminus \operatorname{supp}(u)$. Since I is a matroidal ideal, there exists $j_1 \in \operatorname{supp}(u) \setminus \operatorname{supp}(u_1)$ such that $u_2 = (u_1/e_i)e_{j_1} \in G(I)$. Now we have two following cases:

Case 1: If $j_1 \notin \operatorname{supp}(v_1)$, then

$$j_1 \in \operatorname{supp}(uv) \setminus \operatorname{supp}(u_1v_1) \text{ and } 0 \neq (u_1v_1/e_i)e_{j_1} = u_2v_1 \in G(IJ).$$

So we can choose $j = j_1$.

Case 2: If $j_1 \in \operatorname{supp}(v_1)$, then $j_1 \notin \operatorname{supp}(v)$ since $j_1 \in \operatorname{supp}(u)$ and $uv \neq 0$. So $j_1 \in \operatorname{supp}(v_1) \setminus \operatorname{supp}(v)$. Now since J is matroidal, there exists $k_1 \in \operatorname{supp}(v) \setminus \operatorname{supp}(v_1)$ with $v_2 = (v_1/e_{j_1})e_{k_1} \in G(J)$. Note that $k_1 \neq i$ since $i \notin \operatorname{supp}(v)$ but $k_1 \in \operatorname{supp}(v)$.

If $k_1 \notin \operatorname{supp}(u_2) \setminus \operatorname{supp}(u)$, then $k_1 \notin \operatorname{supp}(u_1)$ since $u_2 = (u_1/e_i)e_{j_1}$. We get

$$k_1 \in \operatorname{supp}(uv) \setminus \operatorname{supp}(u_1v_1)$$

and

$$0 \neq (u_1 v_1/e_i)e_{k_1} = (u_1/e_i)e_{j_1}(v_1/e_{j_1})e_{k_1} = u_2 v_2 \in G(IJ).$$

So we are done because we can choose $j = k_1$.

Otherwise $k_1 \in \operatorname{supp}(u_2) \setminus \operatorname{supp}(u)$. Since I is matroidal, there exists j_2 such that

$$j_2 \in \operatorname{supp}(u) \setminus \operatorname{supp}(u_2) \text{ with } 0 \neq u_3 = (u_2/e_{k_1})e_{j_2} \in G(I).$$

Observe that $j_2 \neq i$ since $j_2 \in \operatorname{supp}(u)$ and $i \notin \operatorname{supp}(u)$. Then we get

$$0 \neq (u_1 v_1/e_i)e_{j_2} = ((u_1/e_i)e_{j_1}/e_{k_1})e_{j_2}(v_1/e_{j_1})e_{k_1} = u_3 v_2 \in G(IJ)$$

and we can choose $j = j_2$. Hence the product of two matroidal ideals is also matroidal.

Now it is obvious that J_i is a matroidal ideal for $i = 1, \ldots, d$. Therefore, J is also a matroidal ideal. So J has a linear resolution by the fact a matroidal ideal has a linear resolution; see Example 3.7.

Note that in the above proof, we need the following lemma:

Lemma 4.3 ([5; Proposition 8.2.17])**.** *Let I be a monomial ideal in the polynomial ring S which is generated in degree d. If I has a d-linear resolution, then the ideal generated by squarefree parts of degree d in I has a d-linear resolution.*

Next we study some further special cases of products of ideals. For this we need the following lemma:

Lemma 4.4. *Let $J \subset E$ be a graded ideal and $f \in E_1$ a linear form such that f is E/J-regular. If J has a d-linear resolution then fJ has a $(d+1)$-linear resolution.*

Proof. By changing the coordinates, we can assume that $f = e_n$ and e_n is E/J-regular. We have $J :_E e_n = J + (e_n)$. Therefore, $J \cap (e_n) = e_n J$. Hence,

$$(J + (e_n))/(e_n) \cong J/(J \cap (e_n)) = J/e_n J.$$

Since J has a d-linear resolution, $(J + (e_n))/(e_n)$ has a d-linear resolution over $E/(e_n) \cong K\langle e_1, \ldots, e_{n-1}\rangle$. Note that the inclusion $K\langle e_1, \ldots, e_{n-1}\rangle \hookrightarrow K\langle e_1, \ldots, e_n\rangle$ is a flat morphism. Therefore, $(J+(e_n))/(e_n)$ also has a d-linear resolution over E, i.e., $\operatorname{reg}(J+(e_n))/(e_n)) = d$.

Now consider the short exact sequence

$$0 \longrightarrow e_n J \longrightarrow J \longrightarrow J/(e_n J) \longrightarrow 0.$$

By Lemma 2.1, we have

$$\operatorname{reg}(e_n J) \leq \max\{\operatorname{reg}(J), \operatorname{reg}(J)/(e_n J) + 1\} = d + 1.$$

Since $e_n J$ is generated in degree $d + 1$, we have $\operatorname{reg}(e_n J) \geq d + 1$. This implies that $\operatorname{reg}(e_n J) = d + 1$.

Considering a product of two or three linear ideals, we have:

Proposition 4.5. *Let I, J be linear ideals such that $IJ \neq 0$. Then IJ has a 2-linear free resolution.*

Proof. Since I, J are linear ideals, we can assume that $I + J = \mathfrak{m}$, otherwise I, J are in a smaller exterior algebra which we can modulo a regular sequence to get $I + J = \mathfrak{m}$. By changing the coordinate and choosing suitable generators, we can assume further that

$$I = (e_1, \dots, e_s) \text{ and } J = (e_{s+1}, \dots, e_n, g_1, \dots, g_r),$$

where $1 \leq s < n$ and g_i is a linear form in $K\langle e_1, \dots, e_s\rangle$ for $i = 1, \dots, r$.

Let $E' = K\langle e_1, \dots, e_{n-1}\rangle$, $J' = (e_{s+1}, \dots, e_{n-1}, g_1, \dots, g_r) \subset E'$ and $I' = (e_1, \dots, e_s) \subset E'$. We have $J = J'E + (e_n)$ and $I = I'E$.

Now we prove the statement by induction on n.

For the case $n = 1$ or $n = 2$, we have only two case $I = (e_1)$ and $J = (e_1)$ or $J = (e_1, e_2)$. Then $IJ = (0)$ or $IJ = (e_1e_2)$, the statement holds for both these cases.

Assume that the statement is true for $n - 1$. This implies that the ideal $I'J'$ has a 2-linear resolution in E', i.e, $\operatorname{reg}_{E'}(I'J') = 2$. Hence, $\operatorname{reg}_E(I'J'E) = 2$. Note that e_n is $I'J'E$-regular. This implies that $IJ' : e_n = IJ' + (e_n)$. Then $IJ' \cap e_nI = e_nIJ'$. In fact, let $f \in IJ' \cap e_nI$, then $f = ge_n$ with $g \in I$. Hence

$$g \in IJ' : e_n = IJ' + (e_n) \text{ and then } e_ng \in e_nIJ'.$$

Therefore, $f \in e_nIJ'$ and we get $IJ' \cap e_nI = e_nIJ'$.

Consider the short exact sequence

$$0 \longrightarrow IJ' \cap e_nI \longrightarrow IJ' \oplus e_nI \longrightarrow IJ' + e_nI \longrightarrow 0.$$

This can be rewritten as

$$0 \longrightarrow e_nIJ' \longrightarrow IJ' \oplus e_nI \longrightarrow IJ \longrightarrow 0.$$

Since $\operatorname{reg}_E(IJ') = 2$ and $\operatorname{reg}_E(e_nIJ') = 3$ by Lemma 4.4, using Lemma 2.1 we get

$$\operatorname{reg}_E(IJ) \leq \max\{\operatorname{reg}_E(IJ'), \operatorname{reg}_E(e_nIJ') - 1\} = 2.$$

It is clear that $\operatorname{reg}_E(IJ) \geq 2$ since IJ is generated in degree 2, so we get $\operatorname{reg}_E(IJ) = 2$. This concludes the proof.

Proposition 4.6. *Let $I, J, P \subset E$ be linear ideals such that $IJP \neq 0$ and*

$$I + J, I + P, J + P \subsetneq I + J + P.$$

Then the product IJP has a 3-linear free resolution.

Proof. Since I, J, P are linear ideals, we can assume that $I + J + P = \mathfrak{m}$ and $I, J, P \subsetneq \mathfrak{m}$. Now we prove the statement by induction on n.

Suppose that the statement holds for $n - 1$, that means for 3 linear ideals in $E' = K\langle e_1, \dots, e_{n-1}\rangle$, their product has a 3-linear free resolution.

Since $I + J \subsetneq \mathfrak{m}$, by changing the coordinate and choosing suitable generators, we can assume that I, J are generated by linear forms in E' and $P = (e_n, f_1, \dots, f_l)$, where $f_i \in E'$ for $i = 1, \dots, l$. Let $P' = (f_1, \dots, f_l)$. We have $IJP = IJP' + e_n IJ$. Since I, J, P' are generated by linear forms in E', by the induction hypothesis and Proposition 4.5, we have that $\operatorname{reg}_{E'}(IJP' \otimes_E E') = 3$ and $\operatorname{reg}_{E'}(IJ \otimes_E E') = 2$. By Lemma 4.4 and the fact that E is a flat extension of E', we get that $\operatorname{reg}_E(e_n IJ) = 3$ and $\operatorname{reg}_E(e_n IJP') = 4$.

Now it is clear that $e_n IJP' \subseteq IJP' \cap e_n IJ$. We aim to prove the equality. Since e_n is E'-regular in E and I, J, P' are generated by linear forms in E', we get that $IJP' : e_n = IJP' + (e_n)$. Let $f \in IJP' \cap e_n IJ$. Then $f = e_n g$ with $g \in IJ$. We have $g \in IJP' : e_n$. This implies that $g \in IJP' + (e_n)$ and then $f = e_n g \in e_n IJP'$. So we get $e_n IJP' = IJP' \cap e_n IJ$. By Lemma 2.1 and the following short exact sequence

$$0 \longrightarrow e_n IJP' \longrightarrow IJP' \oplus (e_n)IJ \longrightarrow IJP \longrightarrow 0,$$

we get

$$\operatorname{reg}_E(IJP) \leq \max\{\operatorname{reg}_E(e_n IJP') - 1, \operatorname{reg}_E(IJP' \oplus e_n IJ)\} = 3.$$

Moreover, IJP is generated in degree 3, so $\operatorname{reg}_E(IJP) \geq 3$. This implies that IJP has a 3-linear free resolution.

Next we consider one more special case of products of ideals: powers of ideals. In [7], Herzog, Hibi and Zheng prove that if a monomial ideal I in the polynomial ring S has a 2-linear resolution, then every power of I has a linear resolution. We have the same result for the exterior algebra:

Proposition 4.7. *Let $J \subset E$ be a nonzero monomial ideal in E. If J has a 2-linear resolution, then every power of J has a linear resolution.*

Proof. Let $I \subset S$ be the ideal in the polynomial ring S corresponding to J. Then I is a squarefree ideal with a 2-linear resolution by [1; Corollary 2.2]. We only need to consider the case $J^m \neq 0$ for an integer m. We have I^m has a linear resolution by [7; Theorem 3.2]. By Lemma 4.3, the squarefree monomial ideal $(I^m)_{[2m]}$ has also a linear resolution. Note that $(I^m)_{[2m]}$ corresponds to J^m in E, so using [1; Corollary 2.2] again, we conclude that J^m has a linear resolution.

Remark 4.8. A linear form f is E/J-regular but it may be not E/J^2-regular. This is a difference between the polynomial ring and the exterior algebra. For instance, let $J = (e_{12} + e_{34}, e_{13}, e_{23})$ in $K\langle e_1, \dots, e_4\rangle$. Then e_4 is E/J-regular since $J : e_4 = J + (e_4)$. But e_4 is not E/J^2-regular since $J^2 = (e_{1234})$ and $J^2 : (e_4) = (e_{123}) + (e_4) \supsetneq J^2 + (e_4)$.

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TÓM TẮT

VỀ IĐÊAN PHÂN BẬC CÓ GIẢI TỰ DO TUYẾN TÍNH VÀ THƯƠNG TUYẾN TÍNH TRONG ĐẠI SỐ NGOÀI

Bài báo này nhằm mục đích nghiên cứu các iđêan phân bậc có giải tự do tuyến tính, có thương tuyến tính trong đại số ngoài. Chúng tôi sử dụng một khái niệm mở rộng của thương tuyến tính, gọi tên là thương tuyến tính từng phần, để đưa ra một chứng minh khác cho một kết quả nổi tiếng rằng một iđêan có thương tuyến tính là tuyến tính từng phần. Sau đó, chúng tôi xét một vài trường hợp đặc biệt mà một tích của các iđêan tuyến tính có một giải tự do tuyến tính.