ON COMPLETE CONVERGENCE FOR WEIGHTED SUMS OF PAIRWISE AND COORDINATEWISE NEGATIVELY DEPENDENT RANDOM VECTORS IN HILBERT SPACES

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Abstract: In this paper, we establish the complete convergence theorem and the Marcinkiewicz–Zymund type strong law of large numbers for sequences of pairwise and coordinatewise negatively dependent random vectors $\{X_n, n \geq 1\}$ taking values in Hilbert spaces.

Keywords: Negative dependence; pairwise negative dependence; Hilbert space; complete convergence; strong law of large numbers.

1 Introduction

The concept of negative dependence was introduced by Lehmann [9] and further investigated by Ebrahimi and Ghosh [6] and Block et al. [3]. A collection of random variables $\{X_1, \ldots, X_n\}$ is said to be negatively dependent (ND) if for all $x_1, \ldots, x_n \in \mathbb{R}$,

$$P(X_1 \leq x_1, \ldots, X_n \leq x_n) \leq P(X_1 \leq x_1) \ldots P(X_n \leq x_n),$$

and

$$P(X_1 > x_1, \ldots, X_n > x_n) \leq P(X_1 > x_1) \ldots P(X_n > x_n).$$

A sequence of random variables $\{X_i, i \geq 1\}$ is said to be ND if for any $n \geq 1$, the collection $\{X_1, \ldots, X_n\}$ is ND. A sequence of random variables $\{X_i, i \geq 1\}$ is said to be pairwise negatively dependent (PND) if for all $x, y \in \mathbb{R}$ and for all $i \neq j$,

$$P(X_i \leq x, X_j \leq y) \leq P(X_i \leq x) P(X_j \leq y). \tag{1.1}$$

It is well known and easy to prove that $\{X_i, i \geq 1\}$ is PND if and only if for all $x, y \in \mathbb{R}$ and for all $i \neq j$,

$$P(X_i > x, X_j > y) \leq P(X_i > x) P(X_j > y).$$

A simple consequence of (1.1) is that if $\{X_1, \ldots, X_n\}$ are PND random variables with finite variances, then for all $i \neq j$, $E(X_i X_j) \leq E X_i E X_j$, and so

$$Var\left(\sum_{i=1}^{n} X_i\right) \leq \sum_{i=1}^{n} Var(X_i). \tag{1.2}$$

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This concept was first extended to Hilbert space-valued random vectors by Burton et al. [4]. After that, a literature of investigation concerning the limit theorems for pairwise and coordinatewise negatively dependent vectors in Hilbert spaces has emerged, including complete convergence (Hien et al. [8]) and weak laws of large numbers (Anh and Hien [1], Dung et al. [5], Hien and Thanh [7]). The notion of pairwise and coordinatewise negatively dependent (PCND) random vectors in Hilbert spaces was introduced by Hien et al. [8]. Let H be a real separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$. A sequence $\{X_n, n \geq 1\}$ of random vectors taking values in H is said to be PCND if for some orthonormal basis $\{e_j, j \in B\}$ and for each $j \in B$, the sequence of random variables $\{\langle X_i, e_j \rangle, i \geq 1\}$ is PND. We would like to note that the PCND structure do not preserve if we change the basis.

This paper aims to establish complete convergence and strong law of large numbers for weighted sums of PCND random vectors taking values in Hilbert spaces. Let $1 \leq p < 2$, $\alpha p \geq 1$, $\{X_n, n \geq 1\}$ be a sequence of PCND identically distributed mean 0 random vectors in H. By techniques developed by Hien et al. [8], we provide the sufficient conditions for

$$\sum_{n=1}^{\infty} n^{\alpha p-2} \mathbb{P}\Big(\max_{1 \leq k \leq n} \Big\| \sum_{i=1}^{k} a_{ni} X_i \Big\| > \varepsilon (n \log^2 n)^{\alpha} \Big) < \infty. \tag{1.3}$$

Throughout this paper, H denotes a real separable Hilbert space with orthonormal basis $\{e_j, j \in B\}$, inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$. The symbol C denotes a generic positive constant whose value may be different for each appearance, and log denotes the logarithm to base 2. By saying $\{X_n, n \geq 1\}$ is a sequence of PCND random vectors, we mean that the random vectors are PCND with respect to the orthonormal basis $\{e_j, j \in B\}$.

2 Main results

Firstly, we present the Rademacher-Menshov type inequality for sums of PCND random vectors in H. This lemma was proved by Hien et al. [8].

Lemma 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of PCND mean 0 random vectors in H satisfying $\mathbb{E}\|X_n\|^2 < \infty$ for all $n \geq 1$. Then for any $n \geq 1$, we have*

$$\mathbb{E}\Big\| \sum_{i=1}^{n} X_i \Big\|^2 \leq \sum_{i=1}^{n} \mathbb{E}\|X_i\|^2, \tag{2.1}$$

and

$$\mathbb{E}\left(\max_{1 \leq k \leq n} \Big\| \sum_{i=1}^{k} X_i \Big\|^2 \right) \leq \log^2(2n) \sum_{i=1}^{n} \mathbb{E}\|X_i\|^2. \tag{2.2}$$

Next, we establish the complete convergence for weighted sums of PCND identically distributed random vectors in Hilbert spaces. Let X be a random vector in H. Here and thereafter, we denote the j^{th} coordinate of X by $X^{(j)}$, i.e.,

$$X^{(j)} = \langle X, e_j \rangle,\ j \in B.$$

Then, we can write

$$X = \sum_{j \in B} X^{(j)} e_j.$$

Theorem 2.2. *Let* $1 \le p < 2$, $\alpha p \ge 1$, $\{X_n, n \ge 1\}$ *be a sequence of PCND identically distributed mean* 0 *random vectors in a finite dimensional* H *and let* $\{a_{ni}, n \ge 1, 1 \le i \le n\}$ *be an array of constants satisfying*

$$\sum_{i=1}^{n} a_{ni}^2 \le Cn, \quad \text{for all } n \ge 1. \tag{2.3}$$

If

$$\mathbb{E}\|X_1\|^p < \infty, \tag{2.4}$$

then for all $\varepsilon > 0$, *we have*

$$\sum_{n=1}^{\infty} n^{\alpha p - 2} \mathbb{P}\Big(\max_{1 \le k \le n} \Big\| \sum_{i=1}^{k} a_{ni} X_i \Big\| > \varepsilon (n \log^2 n)^{\alpha} \Big) < \infty. \tag{2.5}$$

Proof. We can assume that $a_{ni} \ge 0, n \ge 1, 1 \le i \le n$ since in the general case we can use the decomposition $a_{ni} \equiv a_{ni}^{+} - a_{ni}^{-}$. For $n \ge 1, i \ge 1, j \in B$, set

$$\begin{aligned} Y_{ni}^{(j)} &= -(n \log^2 n)^{\alpha} \mathbb{I}\left(X_i^{(j)} < -(n \log^2 n)^{\alpha} \right) \\ &\quad + X_i^{(j)} \mathbb{I}\left(|X_i^{(j)}| \le (n \log^2 n)^{\alpha} \right) \\ &\quad + (n \log^2 n)^{\alpha} \mathbb{I}\left(X_i^{(j)} > (n \log^2 n)^{\alpha} \right), \\ Y_{ni} &= \sum_{j \in B} Y_{ni}^{(j)} e_j, \\ S_{nk} &= \sum_{i=1}^{k} \big(a_{ni} Y_{ni} - \mathbb{E}(a_{ni} Y_{ni}) \big). \end{aligned}$$

For any $\varepsilon > 0$, we have

$$\begin{aligned} &\mathbb{P}\Big(\max_{1 \le k \le n} \Big\| \sum_{i=1}^{k} a_{ni} X_i \Big\| > \varepsilon (n \log^2 n)^{\alpha} \Big) \\ &\le \mathbb{P}\Big(\max_{1 \le k \le n} \Big\| \sum_{i=1}^{k} a_{ni} X_i \Big\| > \varepsilon (n \log^2 n)^{\alpha}; \bigcap_{1 \le i \le n} |X_i^{(j)}| \le (n \log^2 n)^{\alpha} \Big) \\ &\quad + \mathbb{P}\Big(\bigcup_{1 \le i \le n} |X_i^{(j)}| > (n \log^2 n)^{\alpha} \Big) \\ &\le \mathbb{P}\Big(\bigcup_{1 \le i \le n} \bigcup_{j \in B} \big(|X_i^{(j)}| > (n \log^2 n)^{\alpha} \big) \Big) \end{aligned}$$

$$
\begin{aligned}
&\quad + \mathbb{P}\Big(\max_{1\le k\le n} \Big\| \sum_{i=1}^{k} a_{ni} Y_{ni} \Big\| > \varepsilon (n\log^2 n)^{\alpha} \Big) \\
&\le \mathbb{P}\Big(\bigcup_{1\le i\le n} \bigcup_{j\in B} \big(|X_i^{(j)}| > (n\log^2 n)^{\alpha}\big) \Big) \\
&\quad + \mathbb{P}\Big(\max_{1\le k\le n} \Big\| \sum_{i=1}^{k} \mathbb{E}(a_{ni} Y_{ni}) \Big\| > \varepsilon (n\log^2 n)^{\alpha}/2 \Big) \\
&\quad + \mathbb{P}\Big(\max_{1\le k\le n} \|S_{nk}\| > \varepsilon (n\log^2 n)^{\alpha}/2 \Big).
\end{aligned} \tag{2.6}
$$

By (2.4), we have

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2} \mathbb{P}\Big(\bigcup_{1\le i\le n} \bigcup_{j\in B} \big(|X_i^{(j)}| > (n\log^2 n)^{\alpha}\big) \Big) \\
&\le \sum_{n=1}^{\infty} n^{\alpha p-2} \sum_{j\in B} \sum_{i=1}^{n} \mathbb{P}\left(|X_i^{(j)}| > (n\log^2 n)^{\alpha}\right) \\
&\le \sum_{j\in B} \sum_{n=1}^{\infty} n^{\alpha p-2} \sum_{i=1}^{n} \mathbb{P}\left(|X_i^{(j)}| > n^{\alpha}\right) \\
&= \sum_{j\in B} \sum_{n=1}^{\infty} n^{\alpha p-1} \mathbb{P}\left(|X_1^{(j)}| > n^{\alpha}\right) \\
&= \sum_{j\in B} \sum_{n=1}^{\infty} n^{\alpha p-1} \sum_{i=n}^{\infty} \mathbb{P}\left(i^{\alpha} < |X_1^{(j)}| \le (i+1)^{\alpha}\right) \\
&= \sum_{j\in B} \sum_{i=1}^{\infty} \sum_{n=1}^{i} n^{\alpha p-1} \mathbb{P}\left(i^{\alpha} < |X_1^{(j)}| \le (i+1)^{\alpha}\right) \\
&\le C \sum_{j\in B} \sum_{i=1}^{\infty} i^{\alpha p} \mathbb{P}\left(i^{\alpha} < |X_1^{(j)}| \le (i+1)^{\alpha}\right) \\
&\le C \sum_{j\in B} \mathbb{E}|X_1^{(j)}|^p \\
&\le C\mathbb{E}\|X_1\|^p < \infty.
\end{aligned} \tag{2.7}
$$

For $n \ge 1$, by the Cauchy - Schwarz inequality and (2.3),

$$
\Big(\sum_{i=1}^{n} |a_{ni}| \Big)^2 \le n \Big(\sum_{i=1}^{n} a_{ni}^2 \Big) \le Cn^2 \tag{2.8}
$$

and

$$
(n\log^2 n)^{-\alpha} \max_{1\le k\le n} \Big\| \sum_{i=1}^{k} \mathbb{E}(a_{ni} Y_{ni}) \Big\|
$$

$$
\begin{aligned}
&\leq (n\log^2 n)^{-\alpha}\sum_{i=1}^{n}\|\mathbb{E}(a_{ni}Y_{ni})\| \\
&\leq (n\log^2 n)^{-\alpha}\sum_{i=1}^{n}|a_{ni}|\sum_{j\in B}|\mathbb{E}Y_{ni}^{(j)}| \\
&\leq (n\log^2 n)^{-\alpha}\sum_{i=1}^{n}|a_{ni}|\sum_{j\in B}\Big|\mathbb{E}X_i^{(j)}\mathbb{I}\big(|X_i^{(j)}|\leq (n\log^2 n)^{\alpha}\big)\Big| \\
&\quad + (n\log^2 n)^{-\alpha}\sum_{i=1}^{n}|a_{ni}|\sum_{j\in B}(n\log^2 n)^{\alpha}\mathbb{P}\left(|X_i^{(j)}| > (n\log^2 n)^{\alpha}\right) \\
&= (n\log^2 n)^{-\alpha}\sum_{i=1}^{n}|a_{ni}|\sum_{j\in B}\Big|\mathbb{E}X_i^{(j)}\mathbb{I}\big(|X_i^{(j)}| > (n\log^2 n)^{\alpha}\big)\Big| \\
&\quad + \sum_{i=1}^{n}|a_{ni}|\sum_{j\in B}\mathbb{P}\left(|X_i^{(j)}| > (n\log^2 n)^{\alpha}\right) \\
&= Cn^{1-\alpha}\log^{-2\alpha}(n)\sum_{j\in B}\mathbb{E}\left(|X_1^{(j)}|\mathbb{I}\big(|X_1^{(j)}| > (n\log^2 n)^{\alpha}\big)\right) \\
&\quad + C\sum_{j\in B}n\mathbb{P}\left(|X_1^{(j)}| > (n\log^2 n)^{\alpha}\right) \\
&\leq Cn^{1-\alpha}\log^{-2\alpha}(n)\sum_{j\in B}\mathbb{E}\left(|X_1^{(j)}|\mathbb{I}\big(|X_1^{(j)}| > (n\log^2 n)^{\alpha}\big)\right) \\
&\leq Cn^{1-\alpha p}\log^{-2\alpha p}(n)\sum_{j\in B}\mathbb{E}\left(|X_1^{(j)}|^p\mathbb{I}\big(|X_1^{(j)}| > (n\log^2 n)^{\alpha}\big)\right) \\
&\leq Cn^{1-\alpha p}\log^{-2\alpha p}(n)\sum_{j\in B}\mathbb{E}\left(|X_1^{(j)}|^p\right) \to 0, \text{ as } n\to\infty. \qquad (2.9)
\end{aligned}
$$

From (2.6), (2.7) and (2.9), to obtain (2.5), it remains to show that

$$
\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{P}\Big(\max_{1\leq k\leq n}\|S_{nk}\| > \varepsilon(n\log^2 n)^{\alpha}/2\Big) < \infty. \tag{2.10}
$$

It is well known that for all $j\in B$ and for all $n\geq 1$, $\{Y_{ni}^{(j)} - EY_{ni}^{(j)}, i\geq 1\}$ is a sequence of PND random variables. So $\{a_{ni}(Y_{ni} - EY_{ni}), i\geq 1\}$ is a sequence of PCND random vectors in H. By applying Markov's inequality and Lemma 2.1, we have

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{P}\Big(\max_{1\leq k\leq n}\|S_{nk}\| > \varepsilon(n\log^2 n)^{\alpha}/2\Big) \\
&\leq 1 + \sum_{n=2}^{\infty}\frac{4}{\varepsilon^2(n\log^2 n)^{2\alpha}}n^{\alpha p-2}\log^2(2n)\sum_{i=1}^{n}\mathbb{E}\|a_{ni}Y_{ni} - \mathbb{E}(a_{ni}Y_{ni})\|^2
\end{aligned}
$$

$$\leq 1 + C\sum_{n=2}^{\infty} \frac{1}{n^{2+2\alpha-\alpha p}\log^{4\alpha-2} n}\sum_{i=1}^{n}\mathbb{E}\|a_{ni}Y_{ni}\|^2$$
$$\leq 1 + C\sum_{n=2}^{\infty} \frac{1}{n^{2+2\alpha-\alpha p}\log^{4\alpha-2} n}\sum_{i=1}^{n}|a_{ni}|^2\mathbb{E}\|Y_{n1}\|^2$$
$$\leq 1 + C\sum_{j\in B}\sum_{n=2}^{\infty} \frac{1}{n^{1+2\alpha-\alpha p}\log^{4\alpha-2} n}\Bigg(\mathbb{E}\left(|X_1^{(j)}|^2 I\big(|X_1^{(j)}| \leq (n\log^2 n)^{\alpha}\big)\right)$$
$$+ (n\log^2 n)^{2\alpha}\mathbb{P}\left(|X_1^{(j)}| > (n\log^2 n)^{\alpha}\right)\Bigg)$$
$$:= 1 + J_1 + J_2. \tag{2.11}$$

We estimate J_1 as follows

$$J_1 \leq C\sum_{j\in B}\sum_{n=2}^{\infty} \frac{1}{n^{1+2\alpha-\alpha p}\log^{4\alpha-2} n}$$
$$\times \sum_{i=1}^{n-1}\mathbb{E}\left(|X_1^{(j)}|^2\mathbb{I}\big((i\log^2 i)^{\alpha} < |X_1^{(j)}| \leq ((i+1)\log^2(i+1))^{\alpha}\big)\right)$$
$$\leq C\sum_{j\in B}\sum_{i=1}^{\infty}\left(\sum_{n=i+1}^{\infty}\frac{1}{n^{1+2\alpha-\alpha p}\log^{4\alpha-2} n}\right)$$
$$\times \mathbb{E}\left(|X_1^{(j)}|^2\mathbb{I}\big((i\log^2 i)^{\alpha} < |X_1^{(j)}| \leq ((i+1)\log^2(i+1))^{\alpha}\big)\right)$$
$$\leq C\sum_{j\in B}\sum_{i=1}^{\infty}\frac{1}{(i+1)^{2\alpha-\alpha p}\log^{4\alpha-2}(i+1)}$$
$$\times \mathbb{E}\left(|X_1^{(j)}|^2\mathbb{I}\big((i\log^2 i)^{\alpha} < |X_1^{(j)}| \leq ((i+1)\log^2(i+1))^{\alpha}\big)\right)$$
$$\leq C\sum_{j\in B}\sum_{i=1}^{\infty}\log^{2(1-\alpha p)}(i+1)$$
$$\times \mathbb{E}\left(|X_1^{(j)}|^p\mathbb{I}\big((i\log^2 i)^{\alpha} < |X_1^{(j)}| \leq ((i+1)\log^2(i+1))^{\alpha}\big)\right)$$
$$\leq C\sum_{j\in B}\sum_{i=1}^{\infty}\mathbb{E}\left(|X_1^{(j)}|^p\mathbb{I}\big((i\log^2 i)^{\alpha} < |X_1^{(j)}| \leq ((i+1)\log^2(i+1))^{\alpha}\big)\right)$$
$$\leq C\sum_{j\in B}\mathbb{E}|X_1^{(j)}|^p$$
$$\leq C\mathbb{E}\|X_1\|^p < \infty. \tag{2.12}$$

For J_2, we have

$$J_2 \leq C\sum_{j\in B}\sum_{n=2}^{\infty} n^{\alpha p-1}\log^2 n\mathbb{P}\left(|X_1^{(j)}| > (n\log^2 n)^{\alpha}\right)$$

$$
\begin{aligned}
&\le C\sum_{j\in B}\sum_{n=2}^{\infty}\sum_{i=n}^{\infty} n^{\alpha p-1}\log^2 n \\
&\qquad\qquad \times \mathbb{P}\Big((i\log^2 i)^{\alpha} < |X_1^{(j)}| \le ((i+1)\log^2(i+1))^{\alpha}\Big) \\
&\le C\sum_{j\in B}\sum_{i=2}^{\infty}\Big(\sum_{n=2}^{i} n^{\alpha p-1}\log^2 n\Big) \\
&\qquad\qquad \times \mathbb{P}\Big((i\log^2 i)^{\alpha} < |X_1^{(j)}| \le ((i+1)\log^2(i+1))^{\alpha}\Big) \\
&\le C\sum_{j\in B}\sum_{i=2}^{\infty} i^{\alpha p}\log^2 i\,\mathbb{P}\Big((i\log^2 i)^{\alpha} < |X_1^{(j)}| \le ((i+1)\log^2(i+1))^{\alpha}\Big) \\
&\le C\sum_{j\in B}\sum_{i=2}^{\infty}(\log i)^{2(1-\alpha p)} \\
&\qquad\qquad \times \mathbb{E}|X_1^{(j)}|^{p}\mathbb{I}\Big((i\log^2 i)^{\alpha} < |X_1^{(j)}| \le ((i+1)\log^2(i+1))^{\alpha}\Big) \\
&\le C\sum_{j\in B}\mathbb{E}|X_1^{(j)}|^{p} \\
&\le C\mathbb{E}\|X_1\|^{p} < \infty. \qquad (2.13)
\end{aligned}
$$

Combining (2.11), (2.12) and (2.13), we obtain (2.10). The proof is completed. □

The following corollary is the Marcinkiewicz–Zymund type strong law of large number for sequences of PCND random vectors in Hilbert spaces.

Corollary 2.3. *Let* $1 \le p < 2$, $\{X_n, n \ge 1\}$ *be a sequence of PCND identically distributed mean* 0 *random vectors in a finite dimensional* H. *If* (2.4) *holds, then*

$$
\frac{1}{(n\log^2 n)^{1/p}}\sum_{i=1}^{n} X_i \to 0 \ \textit{a.s as}\ n \to \infty.
$$

Proof. For any $\varepsilon > 0$, by applying Theorem 2.2, we have

$$
\begin{aligned}
\infty &> \sum_{n=1}^{\infty} n^{-1}\mathbb{P}\Big(\max_{1\le k\le n}\Big\|\sum_{j=1}^{k} X_j\Big\| > \varepsilon(n\log^2 n)^{1/p}\Big) \\
&= \sum_{i=0}^{\infty}\sum_{n=2^i}^{2^{i+1}-1} n^{-1}\mathbb{P}\Big(\max_{1\le k\le n}\Big\|\sum_{j=1}^{k} X_j\Big\| > \varepsilon(n\log^2 n)^{1/p}\Big) \\
&\ge \sum_{i=0}^{\infty}\sum_{n=2^i}^{2^{i+1}-1} 2^{-i-1}\mathbb{P}\Big(\max_{1\le k\le 2^i}\Big\|\sum_{j=1}^{k} X_j\Big\| > 4\varepsilon(2^i i^2)^{1/p}\Big) \\
&= \frac{1}{2}\sum_{i=0}^{\infty}\mathbb{P}\Big(\max_{1\le k\le 2^i}\Big\|\sum_{j=1}^{k} X_j\Big\| > 4\varepsilon(2^i i^2)^{1/p}\Big). \qquad (2.14)
\end{aligned}
$$

By the Borel-Cantelli lemma, (2.14) ensures that

$$\frac{1}{(2^i i^2)^{1/p}} \max_{1\le k\le 2^i} \left\| \sum_{j=1}^{k} X_j \right\| \to 0 \text{ a.s, as } i \to \infty. \tag{2.15}$$

For $2^{i-1} \le n < 2^i$, we have

$$0 \le \frac{1}{(n\log^2 n)^{1/p}} \left\| \sum_{j=1}^{n} X_j \right\| \le \frac{4}{(2^i i^2)^{1/p}} \max_{1\le k\le 2^i} \left\| \sum_{j=1}^{k} X_j \right\|. \tag{2.16}$$

The conclusion of the corollary follows from (2.15) and (2.16). □

Remark 2.4. (i) We see that Theorem 2.2 is an extension of Theorem 3.3 of Hien et al. [8], by letting $\alpha p = 1$ and $a_{ni} \equiv 1$.
(ii) Theorem 2.2 can be viewed as a Baum-Katz type theorem [2] for PCND random vectors in Hilbert spaces, by letting $\alpha p = 1$ and $a_{ni} \equiv 1$.

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TÓM TẮT

SỰ HỘI TỤ ĐẦY ĐỦ CỦA TỔNG CÓ TRỌNG SỐ CỦA CÁC PHẦN TỬ NGẪU NHIÊN PHỤ THUỘC ÂM ĐÔI MỘT THEO TỌA ĐỘ NHẬN GIÁ TRỊ TRONG KHÔNG GIAN HILBERT

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Trong bài báo này, chúng tôi thiết lập sự hội tụ đầy đủ và luật mạnh số lớn dạng Marcinkiewicz–Zymund cho các phần tử ngẫu nhiên phụ thuộc âm đôi một theo tọa độ $\{X_n, n \geq 1\}$ nhận giá trị trong không gian Hilbert.

Từ khóa: Phụ thuộc âm; phụ thuộc âm đôi một; không gian Hilbert; hội tụ đầy đủ; luật mạnh số lớn.