SOME LAWS OF LARGE NUMBERS FOR DOUBLE ARRAYS OF RANDOM UPPER SEMICONTINUOUS FUNCTIONS

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Received on 20/4/2021, accepted for publication on 30/6/2021

Abstract: In this paper, we introduce some laws of large numbers for double array of level-wise negatively associated random upper semicontinuous functions under various settings. We also establish some maximal inequalities for 2-dimensional structure. Our results are extensions for corresponding ones in the literature.

Keywords: Random set; level-wise negatively associated; random upper semicontinuous functions.

1 Introduction

The notion of dependence is negatively associated which was first introduced in 1981 by Alam and Saxena [1], and carefully studied by Joag-Dev and Proschan [3] in 1983. In the last years, there has been growing interest in concepts of negative association for families of random variables because of their wide applications in multivariate statistical analysis, reliability theory, percolation, and statistical physics. In this paper, we consider the above dependence notion in the space of upper semicontinuous functions with multidimensional indices.

Random upper semicontinuous functions were introduced to the model of random elements that take values being upper semicontinuous functions (see [2, 8, 10]). Limit theorems for the class of random upper semicontinuous functions have received much attention because of their usefulness in several applied fields, especially, one of them is the law of large numbers. The laws of large numbers for random upper semicontinuous functions (or fuzzy random sets) have been studied by many authors such as Joo and Kim [4], N. V. Quang and D. X. Giap [7], N. T. Thuan and N. V. Quang [10], etc.

The concepts of negatively associated, negatively dependent for random upper semicontinuous functions were presented by N. T. Thuan and N. V. Quang [10] and they also obtained some laws of large numbers for the class of these dependent functions with the sequence structure. The aim of this paper is to prove some laws of large numbers for double arrays of negatively associated random upper semicontinuous functions under various settings. To obtain these limited results, we have to establish some maximal inequalities for double arrays of negatively associated random variables and negatively associated random upper semicontinuous functions.

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2 Preliminaires

Throughout this paper, let $(\Omega, \mathcal{A}, \mathbf{P})$ be a complete probability space. In the present paper, $\mathbb{R}$ (resp. $\mathbb{N}$) will denote the set of all real numbers (resp. positive integers). In this section, we sum up some basic notions and related properties for random upper semicontinuous functions which were presented in [2, 5, 10].

Let $\mathcal{K}$ be the set of compact intervals of $\mathbb{R}$. If x is an element of $\mathcal{K}$ then it will be denoted by $x = [x^{(1)}; x^{(2)}]$, where $x^{(1)}, x^{(2)}$ are two end points. The Hausdorff distance d_H on $\mathcal{K}$ is defined by

$$d_H(x, y) = \max\{|x^{(1)} - y^{(1)}|; |x^{(2)} - y^{(2)}|\}, x, y \in \mathcal{K}.$$

A linear structure in $\mathcal{K}$ is defined as follows:

$$x + y = [x^{(1)} + y^{(1)}; x^{(2)} + y^{(2)}],$$

$$\lambda x = \begin{cases} [\lambda x^{(1)}; \lambda x^{(2)}] & \text{if } \lambda \geq 0 \\ [\lambda x^{(2)}; \lambda x^{(1)}] & \text{if } \lambda < 0. \end{cases}$$

For a function $u : \mathbb{R} \to [0; 1]$, the *$\alpha$-level set* of u is defined by $[u]_\alpha = \{x \in \mathbb{R} : u(x) \geq \alpha\}$ for each $\alpha \in (0; 1]$. For each $\alpha \in [0; 1)$, $[u]_{\alpha+}$ denotes the closure of $\{x \in \mathbb{R} : u(x) > \alpha\}$. In particular, $[u]_{0+}$ is called the *support* of u and denoted by supp$u$. The function $u : \mathbb{R} \to [0; 1]$ is called *upper semicontinuous function* if only if $[u]_\alpha$ is the closed set for all $\alpha \in (0, 1]$. A upper semicontinuous function $u : \mathbb{R} \to [0; 1]$ is called *quasiconcave function* if $u(\lambda x + (1 - \lambda)y) \geq \min\{u(x), u(y)\}$ for all $x, y \in \mathbb{R}$, and its equivalent condition is that $[u]_\alpha$ is a convex subset of $\mathbb{R}$ for every $\alpha \in (0; 1]$. Let $\mathcal{U}$ denote the family of all upper semicontinuous functions $u : \mathbb{R} \to [0; 1]$ satisfying the following conditions

(i) supp u is compact;

(ii) $[u]_1 \neq \emptyset$;

(iii) u is quasiconcave.

Therefore, if $u \in \mathcal{U}$ then for each $\alpha \in (0; 1]$, $[u]_\alpha$ is an interval of $\mathbb{R}$ and denoted by $[u]_\alpha = [[u]_\alpha^{(1)}; [u]_\alpha^{(2)}]$, where $[u]_\alpha^{(1)}$ and $[u]_\alpha^{(2)}$ are two end points of the interval.

The addition and scalar multiplication on $\mathcal{U}$ are defined by

$$(u + v)(x) = \sup_{y+z=x} \min\{u(y), v(z)\}, \ (\lambda u)(x) = \begin{cases} u(\lambda^{-1}x) & \text{if } \lambda \neq 0 \\ \widetilde{0} & \text{if } \lambda = 0, \end{cases}$$

where $u, v \in \mathcal{U}, \lambda \in \mathbb{R}$ and $\widetilde{0} = I_{\{0\}}$ is the indicator function of $\{0\}$. Then for $u, v \in \mathcal{U}, \lambda \in \mathbb{R}$ we have $[u + v]_\alpha = [u]_\alpha + [v]_\alpha$ and $[\lambda u]_\alpha = \lambda [u]_\alpha$ for each $\alpha \in (0; 1]$.

The following metrics on $\mathcal{U}$ are often used: for $u, v \in \mathcal{U}$

$$D_\infty(u, v) = \sup_{\alpha \in (0;1]} d_H([u]_\alpha, [v]_\alpha) = \sup_{\alpha \in [0;1]} d_H([u]_\alpha, [v]_\alpha).$$

It is known that the metric space $(\mathcal{U}, D_\infty)$ is complete but not separable (see [5]). For $u \in \mathcal{U}$, denote $\|u\|_\infty = D_\infty(u, \widetilde{0})$.

As introduced in [2], the mapping $\langle .,. \rangle : \mathcal{K} \times \mathcal{K} \to \mathbb{R}$ is defined by

$$\langle a, b\rangle = \frac{1}{2}\big(a^{(1)}b^{(1)} + a^{(2)}b^{(2)}\big), \text{ where } a = [a^{(1)}, a^{(2)}], b = [b^{(1)}, b^{(2)}],$$

and another metric d_* on $\mathcal{K}$ is defined by

$$d_*(a,b) = \big(\langle a,a\rangle - 2\langle a,b\rangle + \langle b,b\rangle\big)^{1/2} = \left(\frac{1}{2}\Big(\big(a^{(1)} - b^{(1)}\big)^2 + \big(a^{(2)} - b^{(2)}\big)^2\Big)\right)^{1/2}.$$

It is easy to check that the metric space $(\mathcal{K}, d_*)$ is complete and separable (see [10]).

As in [2], define $\langle .,. \rangle : \mathcal{U} \times \mathcal{U} \to \mathbb{R}$ by

$$\langle u, v\rangle = \int_0^1 \langle [u]_\alpha, [v]_\alpha\rangle d\alpha.$$

For $u, v \in \mathcal{U}$, denote

$$D_*(u,v) = \Big(\langle u,u\rangle - 2\langle u,v\rangle + \langle v,v\rangle\Big)^{1/2} = \left(\int_0^1 d_*^2\big([u]_\alpha, [v]_\alpha\big) d\alpha\right)^{1/2}.$$

It is clear that D_* is a metric on $\mathcal{U}$ and we also deduce that the metric space $(\mathcal{U}, D_*)$ is separable but not complete. For $u \in \mathcal{U}$, denote $\|u\|_* = D_*(u, \widetilde{0})$.

A mapping $X : \Omega \to \mathcal{K}$ is called a *$\mathcal{K}$-valued random variable* if $X^{-1}(B) \in \mathcal{A}$ for all $B \in \mathcal{B}(\mathcal{K})$, where $\mathcal{B}(\mathcal{K})$ is the Borel σ-algebra on $(\mathcal{K}, d_H)$.

A mapping $X : \Omega \to \mathcal{U}$ is called a *$\mathcal{U}$-valued random variable* (or *random upper semicontinuous function*) if $[X]_\alpha$ $[X]_\alpha$ is a $\mathcal{K}$-valued random variable for all $\alpha \in (0;1]$.

For $p > 0$, denote by $\mathcal{L}^p_@(\mathcal{U})$ the class of $\mathcal{U}$-valued random variables X satisfying $E\|X\|^p_@ < \infty$ (where the symbol @ represents the $*$, ∞). If $X \in \mathcal{L}^1_\infty(\mathcal{U})$ then X is said to be *D_∞-integrable*, this implies that $[X]^{(1)}_\alpha$ and $[X]^{(2)}_\alpha$ are integrable real-valued random variables for all $\alpha \in (0;1]$.

Let X be a D_∞-integrable $\mathcal{U}$-valued random variable. Then, the *expectation* of X, denoted by EX, is defined as an upper semicontinuous function whose α-level set $[EX]_\alpha$ is given by

$$[EX]_\alpha = \big[[EX]^{(1)}_\alpha; [EX]^{(2)}_\alpha\big] = \big[E[X]^{(1)}_\alpha; E[X]^{(2)}_\alpha\big]$$

for each $\alpha \in (0;1]$.

For $X, Y \in \mathcal{L}^1_\infty(\mathcal{U}) \cap \mathcal{L}^2_*(\mathcal{U})$, the notions of variance of X and covariance of X, Y were introduced in [2] as follows:

$$\mathrm{Cov}(X,Y) = E\langle X, Y\rangle - \langle EX, EY\rangle,$$
$$\mathrm{Var}X = \mathrm{Cov}(X,X) = E\langle X, X\rangle - \langle EX, EX\rangle.$$

We also obtain the following property of the variance (see [2])

$$\text{Var}X = \frac{1}{2}\int_0^1 \Big(\text{Var}\big([X]_\alpha^{(1)}\big) + \text{Var}\big([X]_\alpha^{(2)}\big)\Big)d\alpha = ED_*^2(X, EX).$$

For $a, b \in \mathbb{R}$, $\max\{a, b\}$ will be denoted by $a \vee b$ and $\min\{a, b\}$ will be denoted by $a \wedge b$.

For convenience, from now until the end of the paper, we use $S_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}$, $m \geq 1$, $n \geq 1$.

Let $\{b_{mn}, m \geq 1, n \geq 1\}$ be a 2-dimensional array of real numbers. We define

$$\Delta b_{mn} := b_{mn} - b_{m,n-1} - b_{m-1,n} + b_{m-1,n-1}, \text{ for every } m \geq 1, n \geq 1,$$

with the convention that $b_{mn} = 0$ if $m.n = 0$.

3 Main Results

Definition 3.1. (1) A finite family $\{X_1, X_2, \ldots, X_n\}$ of real-valued random variables is said to be *negatively associated* if for any disjoint subsets A_1, A_2 of $\{1, 2, \ldots, n\}$ and any real coordinatewise nondecreasing functions f_1 on $\mathbb{R}^{|A_1|}$, f_2 on $\mathbb{R}^{|A_2|}$, then

$$\text{Cov}\big[f_1(X_i, i \in A_1), f_2(X_j, j \in A_2)\big] \leq 0,$$

whenever the covariance exists, where $|A|$ denotes the cardinality of A.

An infinite family of random variables is negatively associated if every finite subfamily is negatively associated.

(2) Let $\{X_{mn}, m \geq 1, n \geq 1\}$ be a double array of $\mathcal{K}$-valued random variables . Then, $\{X_{mn}, m \geq 1, n \geq 1\}$ is said to be *negatively associated* if $\{X_{mn}^{(1)}, m \geq 1, n \geq 1\}$ and $\{X_{mn}^{(2)}, m \geq 1, n \geq 1\}$ are double arrays of negatively associated real-valued random variables.

(3) Let $\{X_{mn}, m \geq 1, n \geq 1\}$ be a double array of $\mathcal{U}$-valued random variables. Then $\{X_{mn}, m \geq 1, n \geq 1\}$ is said to be *level-wise negatively associated* if $\{[X_{mn}]_\alpha, m \geq 1, n \geq 1\}$ are double arrays of negatively associated $\mathcal{K}$-valued random variables for all $\alpha \in (0; 1]$.

Now we proceed to state our main results. At first, we need some results which will be used later.

Proposition 3.2. *Let $\{X_{mn}, m \geq 1, n \geq 1\}$ be a double array of negatively associated real-valued random variables with $EX_{mn} = 0$ và $EX_{mn}^2 < \infty$, $m \geq 1, n \geq 1$. Then there exists a positive constant C such that for every $m \geq 1, n \geq 1$*

$$E\Big(\max_{k\leq m, l\leq n}\big|\sum_{i=1}^{k}\sum_{j=1}^{l} X_{ij}\big|\Big) \leq C\Big(\sum_{i=1}^{m}\sum_{j=1}^{n} EX_{ij}^2\Big)^{1/2}. \tag{3.1}$$

Proof. The proof is based on a good idea of Utev and Peligrad (see [11, Proposition 3.1]. For each $m \geq 1, n \geq 1$, define

$$a_{mn} = \sup_X \left(E\Big(\max_{k \leq m, l \leq n} | \sum_{i=1}^{k} \sum_{j=1}^{l} X_{ij} | \Big) \Big/ \Big(\sum_{i=1}^{m} \sum_{j=1}^{n} EX_{ij}^2 \Big)^{1/2} \right), \tag{3.2}$$

where the supremum is taken over all fields $X := \{X_{ij}\}$ of square-integrable centered negatively associated random variables with $\sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} EX_{ij}^2 > 0$.

Fix such a random field $\{X_{ij}\}$ and also without loss of generality assume that

$$\sum_{i=1}^{m} \sum_{j=1}^{n} EX_{ij}^2 = 1.$$

Let M be a positive integer that will be specified later. Let $f(x) = ((-M^{-1/2}) \vee x) \wedge M^{-1/2}$, i.e. $f(x) = -M^{-1/2}.I_{(x<-M^{-1/2})} + x.I_{(|x| \leq M^{-1/2})} + M^{-1/2}.I_{(x>M^{-1/2})}$. For $1 \leq i \leq m$, $1 \leq j \leq n$ define: $\xi_{ij} = f(X_{ij}) - Ef(X_{ij})$ and $\eta_{ij} = X_{ij} - \xi_{ij}$. Then both $\{\xi_{ij}\}$ and $\{\eta_{ij}\}$ are double arrays of negatively associated real-valued random variables. Since

$$\begin{aligned}
E|\eta_{ij}| &= E|(X_{ij} - f(X_{ij})) - E(X_{ij} - f(X_{ij}))| \quad (\text{by } EX_{ij} = 0) \\
&\leq 2E|X_{ij} - f(X_{ij})| \\
&= 2E\Big|(X_{ij} + M^{-1/2}).I_{(X_{ij}<-M^{-1/2})} + (X_{ij} - M^{-1/2}).I_{(X_{ij}>M^{-1/2})}\Big| \\
&= 2E\Big(|X_{ij} + M^{-1/2}|.I_{(X_{ij}<-M^{-1/2})} + |X_{ij} - M^{-1/2}|.I_{(X_{ij}>M^{-1/2})}\Big) \\
&\leq 2E\Big(|X_{ij}|.I_{(X_{ij}<-M^{-1/2})} + |X_{ij}|.I_{(X_{ij}>M^{-1/2})}\Big) \\
&= 2E\left|X_{ij}.I_{(|X_{ij}|>M^{-1/2})}\right| \\
&\leq 2M^{1/2}EX_{ij}^2.I_{(|X_{ij}|>M^{-1/2})} \\
&\leq 2M^{1/2}EX_{ij}^2.
\end{aligned}$$

This implies that

$$\sum_{i=1}^{m} \sum_{j=1}^{n} E|\eta_{ij}| \leq 2M^{1/2} \sum_{i=1}^{m} \sum_{j=1}^{n} EX_{ij}^2 = 2M^{1/2}.$$

Therefore,

$$E \max_{k \leq m, l \leq n} \left| \sum_{i=1}^{k} \sum_{j=1}^{l} X_{ij} \right| \leq E \max_{k \leq m, l \leq n} \left| \sum_{i=1}^{k} \sum_{j=1}^{l} \xi_{ij} \right| + 2M^{1/2}.$$

To estimate $E \max\limits_{k \leq m, l \leq n} \left| \sum\limits_{i=1}^{k} \sum\limits_{j=1}^{l} \xi_{ij} \right|$, we shall use a blocking procedure.

Take $u_0 = 0$ and define the integers u_t recursively by

$$u_t = \min\Big\{u : u > u_{t-1}, \sum_{j=u_{t-1}+1}^{u} \sum_{i=1}^{m} E\xi_{ij}^2 > \frac{1}{M}\Big\}.$$

Note that, if we denote by s the number of integers produced by this procedure, we have

$$\frac{s-1}{M} < \sum_{t=1}^{s-1} \sum_{j=u_{t-1}+1}^{u_t} \sum_{i=1}^{m} E\xi_{ij}^2 \leq 1 \text{ so that } s \leq M.$$

Write $S'_{kt} = \sum\limits_{j=u_{t-1}+1}^{u_t} \sum\limits_{i=1}^{k} \xi_{ij}$ for $1 \leq t \leq s-1$ and for convenience, $u_s = n$ that is $S'_{ks} = \sum\limits_{j=u_{s-1}+1}^{n} \sum\limits_{i=1}^{k} \xi_{ij}$. If $v = u_t$ for $1 \leq t \leq s$ then

$$\Big|\sum_{i=1}^{k}\sum_{j=1}^{l} \xi_{ij}\Big| = \Big|\sum_{j=1}^{t} S'_{kj}\Big|,$$

and if $u_{t-1} < v < u_t$ for $1 \leq t \leq s$ then

$$\begin{aligned}\Big|\sum_{i=1}^{k}\sum_{j=1}^{l} \xi_{ij}\Big| &\leq \Big|\sum_{j=1}^{t-1} S'_{kj}\Big| + \Big|\sum_{p=u_{t-1}+1}^{v} \sum_{i=1}^{k} \xi_{ip}\Big| \\ &\leq \Big|\sum_{j=1}^{t-1} S'_{kj}\Big| + \max_{u_{t-1}<v<u_t} \Big|\sum_{p=u_{t-1}+1}^{v} \sum_{i=1}^{k} \xi_{ip}\Big|.\end{aligned}$$

This implies that

$$\begin{aligned}&E \max_{k\leq m, l\leq n} \Big|\sum_{i=1}^{k}\sum_{j=1}^{l} \xi_{ij}\Big| \\ \leq\;& E \max_{1\leq t\leq s} \max_{k\leq m} \Big|\sum_{j=1}^{t} S'_{kj}\Big| + E \max_{1\leq t\leq s} \max_{k\leq m} \left(\max_{u_{t-1}<v<u_t} \Big|\sum_{p=u_{t-1}+1}^{v} \sum_{i=1}^{k} \xi_{ip}\Big|\right) \\ :=\;& \mathrm{I}_1 + \mathrm{I}_2.\end{aligned}$$

For I_1, we have

$$\mathrm{I}_1 \leq \sum_{t=1}^{s} E \max_{k\leq m} \big|S'_{kt}\big| = \sum_{t=1}^{s} E \max_{k\leq m} \Big|\sum_{i=1}^{k} \Big(\sum_{j=u_{t-1}+1}^{u_t} \xi_{ij}\Big)\Big|.$$

Put $Y_i = \sum\limits_{j=u_{t-1}+1}^{u_t} \xi_{ij}$, since $\{\xi_{ij}\}$ is a double array of mean zero negatively associated real-valued random variables then for each t, $\{Y_i\}$ is also a sequence of mean zero negatively associated real-valued random variables. By applying [9, the inequality (1.6) of Theorem 2] for $p = 2$ and by Liapunov's inequality, we obtain the inequality (3.1) for the case of the sequence of random variables with $C = \sqrt{2}$. Combine this with the negative association of the sequence $\{\xi_{ij} : u_{t-1} + 1 \le j \le u_t\}$ of mean zero random variables, we obtain

$$\begin{aligned} E \max_{k \le m} \left| S'_{kt} \right| &= E \max_{k \le m} \Big| \sum_{i=1}^{k} Y_i \Big| \\ &\le \sqrt{2} \Big(\sum_{i=1}^{m} E Y_i^2 \Big)^{1/2} \\ &= \sqrt{2} \Big(\sum_{i=1}^{m} E \Big(\sum_{j=u_{t-1}+1}^{u_t} \xi_{ij} \Big)^2 \Big)^{1/2} \\ &\le \sqrt{2} \Big(\sum_{i=1}^{m} \sum_{j=u_{t-1}+1}^{u_t} E \xi_{ij}^2 \Big)^{1/2}. \end{aligned}$$

Moreover, by the definition of u_t we have

$$\sum_{j=u_{t-1}+1}^{u_t} \sum_{i=1}^{m} E \xi_{ij}^2 > \frac{1}{M} \text{ so that } M^{1/2} \left(\sum_{j=u_{t-1}+1}^{u_t} \sum_{i=1}^{m} E \xi_{ij}^2 \right)^{1/2} > 1.$$

Therefore,

$$\sum_{t=1}^{s} \left(\sum_{j=u_{t-1}+1}^{u_t} \sum_{i=1}^{m} E \xi_{ij}^2 \right)^{1/2} \le M^{1/2} \sum_{t=1}^{s} \sum_{j=u_{t-1}+1}^{u_t} \sum_{i=1}^{m} E \xi_{ij}^2 \le M^{1/2} \sum_{i=1}^{m} \sum_{j=1}^{n} E \xi_{ij}^2.$$

Hence, $\mathrm{I}_1 \le \sqrt{2} M^{1/2}$.
For I_2, we obtain

$$\begin{aligned} (\mathrm{I}_2)^4 &\le E \max_{1 \le t \le s} \max_{k \le m} \left(\max_{u_{t-1} < v < u_t} \Big| \sum_{p=u_{t-1}+1}^{v} \sum_{i=1}^{k} \xi_{ip} \Big|^4 \right) \\ &\le \sum_{t=1}^{s} E \max_{k \le m} \max_{u_{t-1} < v < u_t} \Big| \sum_{p=u_{t-1}+1}^{v} \sum_{i=1}^{k} \xi_{ip} \Big|^4. \end{aligned}$$

On the other hand, by applying the Lemma A.2 [12] in the case $d = 2$ and $p = 4$, we get

$$\begin{aligned} & E \max_{k \le m} \max_{u_{t-1} < v < u_t} \Big| \sum_{p=u_{t-1}+1}^{v} \sum_{i=1}^{k} \xi_{ip} \Big|^4 \\ \le \ & C \Big\{ \Big(E \max_{k \le m} \max_{u_{t-1} < v < u_t} \Big| \sum_{p=u_{t-1}+1}^{v} \sum_{i=1}^{k} \xi_{ip} \Big| \Big)^4 \end{aligned}$$

$$+\sum_{p=u_{t-1}+1}^{u_t-1}\sum_{i=1}^{m}E\xi_{ip}^4+\Big(\sum_{p=u_{t-1}+1}^{u_t-1}\sum_{i=1}^{m}E\xi_{ip}^2\Big)^2\Bigg\}.$$

By the definition of ξ_{ij}, u_t and by using the notation (3.2) for a_{mn}, we obtain

$$\sum_{p=u_{t-1}+1}^{u_t-1}\sum_{i=1}^{m}E\xi_{ip}^2\le\frac{1}{M},$$

$$\Big(E\max_{k\le m}\max_{u_{t-1}<v<u_t}\Big|\sum_{p=u_{t-1}+1}^{v}\sum_{i=1}^{k}\xi_{ip}\Big|\Big)^4\le a_{mn}^4\left(\sum_{p=u_{t-1}+1}^{u_t-1}\sum_{i=1}^{m}E\xi_{ip}^2\right)^2\le\frac{a_{mn}^4}{M^2},$$

and

$$\sum_{p=u_{t-1}+1}^{u_t-1}\sum_{i=1}^{m}E\xi_{ip}^4\le\frac{4}{M}\sum_{p=u_{t-1}+1}^{u_t-1}\sum_{i=1}^{m}E\xi_{ip}^2\le\frac{4}{M^2}.$$

Hence

$$(\mathrm{I}_2)^4\le C\sum_{t=1}^{s}\Big(\frac{a_{mn}^4}{M^2}+\frac{4}{M^2}+\frac{1}{M^2}\Big)\le C\frac{a_{mn}^4+5}{M}.$$

Combining above estimates for I_1 and I_2, we get

$$\begin{aligned}E\max_{k\le m,l\le n}\Big|\sum_{i=1}^{k}\sum_{j=1}^{l}X_{ij}\Big| &\le E\max_{k\le m,l\le n}\Big|\sum_{i=1}^{k}\sum_{j=1}^{l}\xi_{ij}\Big|+2M^{1/2}\\ &\le \sqrt{2}M^{1/2}+\Big(\frac{C}{M}\Big)^{1/4}a_{mn}+\Big(\frac{5C}{M}\Big)^{1/4}+2M^{1/2}\\ &\le \Big(\frac{C}{M}\Big)^{1/4}a_{mn}+\Big(\frac{5C}{M}\Big)^{1/4}+(\sqrt{2}+2)M^{1/2}.\end{aligned}$$

Therefore, by the definition of a_{mn},

$$a_{mn}\le\Big(\frac{C}{M}\Big)^{1/4}a_{mn}+\Big(\frac{5C}{M}\Big)^{1/4}+(\sqrt{2}+2)M^{1/2}.$$

Letting $M=[16C]+1$ yields

$$a_{mn}\le 2+2(\sqrt{2}+2)M^{1/2}.$$

The proof is now completed. □

Remark 3.3. The result of Proposition 3.2 was also introduced in [L. X. Zhang (2006), Maximal inequalities and a law of the iterated logarithm for negatively associated random fields, *arXiv:math/0610511 [math.PR]*]. However, in L. X. Zhang's proof, some specific estimates seem not so clear. In this paper, we demonstrate the Zhang's result in more detail.

Combining the Lemma A.2 [12] and (3.1) we obtain the Rosenthal's type moment inequality for the maximum partial sums.

Proposition 3.4. *Let $p \geq 2$ and let $\{X_{mn}, m \geq 1, n \geq 1\}$ be a double array of negatively associated real-valued random variables with $EX_{mn} = 0$ and $EX_{mn}^2 < \infty$, $m \geq 1, n \geq 1$. Then there exists a constant C_p depending only on p such that for every $m \geq 1, n \geq 1$, we have*

$$E \max_{k \leq m, l \leq n} |\sum_{i=1}^{k} \sum_{j=1}^{l} X_{ij}|^p \leq C_p \Big\{ \Big(\sum_{i=1}^{m} \sum_{j=1}^{n} EX_{ij}^2 \Big)^{p/2} + \sum_{i=1}^{m} \sum_{j=1}^{n} E|X_{ij}|^p \Big\}. \qquad (3.3)$$

In 2000, Q. M. Shao [9] established inequality (3.3) for the case of the sequence of negatively associated random variables.

The following theorem establishes Hájek-Rényi's type maximal inequality for an array of level-wise negatively associated $\mathcal{U}$-valued random variables and this result is obtained in the setting with respect to metric D_*. It plays the key role to derive the laws of large numbers.

Theorem 3.5. *Let $\{b_{mn}, m \geq 1, n \geq 1\}$ be a 2-dimensional array of positive real numbers with $\Delta b_{mn} \geq 0$ for all $m \geq 1, n \geq 1$. Assume that $\{X_{mn}, m \geq 1, n \geq 1\}$ is a double array of D_∞-integrable, level-wise negatively associated $\mathcal{U}$-valued random variables with $E\|X_{mn}\|_*^2 < \infty$, $m \geq 1, n \geq 1$. Then, there exists a positive constant C such that for any $\varepsilon > 0$ and for any $1 \leq s \leq m$, $1 \leq t \leq n$,*

$$\mathbf{P}\Big(\max_{s \leq k \leq m, t \leq l \leq n} \frac{1}{b_{kl}} D_*(S_{kl}, ES_{kl}) \geq \varepsilon \Big) \leq \frac{C}{\varepsilon^2} \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{\mathrm{Var} X_{ij}}{(b_{ij} + b_{st})^2}. \qquad (3.4)$$

Proof. By Markov's inequality, we have

$$\begin{aligned}
& \mathbf{P}\Big(\max_{s \leq k \leq m, t \leq l \leq n} \frac{1}{b_{kl}} D_*(S_{kl}, ES_{kl}) \geq \varepsilon \Big) \\
= \; & \mathbf{P}\Big(\max_{s \leq k \leq m, t \leq l \leq n} \frac{1}{2b_{kl}} D_*(S_{kl}, ES_{kl}) \geq \frac{\varepsilon}{2} \Big) \\
\leq \; & \mathbf{P}\Big(\max_{s \leq k \leq m, t \leq l \leq n} \frac{1}{b_{st} + b_{kl}} D_*(S_{kl}, ES_{kl}) \geq \frac{\varepsilon}{2} \Big) \\
\leq \; & \mathbf{P}\Big(\max_{1 \leq k \leq m, 1 \leq l \leq n} \frac{1}{b_{st} + b_{kl}} D_*(S_{kl}, ES_{kl}) \geq \frac{\varepsilon}{2} \Big) \\
\leq \; & \frac{4}{\varepsilon^2} E\Big(\max_{1 \leq k \leq m, 1 \leq l \leq n} \frac{1}{b_{st} + b_{kl}} D_*(S_{kl}, ES_{kl}) \Big)^2 \\
= \; & \frac{4}{\varepsilon^2} E\Big(\max_{1 \leq k \leq m, 1 \leq l \leq n} \frac{1}{(b_{st} + b_{kl})^2} D_*^2(S_{kl}, ES_{kl}) \Big).
\end{aligned}$$

By putting $r_{kl} = b_{st} + b_{kl}$, we derive

$$\frac{1}{(b_{st} + b_{kl})^2} D_*^2(S_{kl}, ES_{kl}) = \frac{1}{(r_{kl})^2} \int_0^1 d_*^2\Big([S_{kl}]_\alpha, [ES_{kl}]_\alpha \Big) d\alpha$$

$$= \frac{1}{2r_{kl}^2} \int_0^1 \Big(\big([S_{kl}]_\alpha^{(1)} - [ES_{kl}]_\alpha^{(1)} \big)^2 + \big([S_{kl}]_\alpha^{(2)} - [ES_{kl}]_\alpha^{(2)} \big)^2 \Big) d\alpha$$

$$= \frac{1}{2}\int_0^1 \left(\left| \frac{\sum_{i=1}^{k}\sum_{j=1}^{l}\left([X_{ij}]_\alpha^{(1)} - E[X_{ij}]_\alpha^{(1)}\right)}{r_{kl}} \right|^2 + \left| \frac{\sum_{i=1}^{k}\sum_{j=1}^{l}\left([X_{ij}]_\alpha^{(1)} - E[X_{ij}]_\alpha^{(1)}\right)}{r_{kl}} \right|^2 \right) d\alpha.$$

Moreover, for each $p = 1, 2$

$$\begin{aligned}
&\sum_{i=1}^{k}\sum_{j=1}^{l}\left([X_{ij}]_\alpha^{(p)} - E[X_{ij}]_\alpha^{(p)}\right) \\
&= \sum_{i=1}^{k}\sum_{j=1}^{l}\left(\sum_{u=1}^{i}\sum_{v=1}^{j}\Delta r_{uv}\right)\frac{[X_{ij}]_\alpha^{(p)} - E[X_{ij}]_\alpha^{(p)}}{r_{ij}} \\
&= \sum_{u=1}^{k}\sum_{v=1}^{l}\Delta r_{uv}\left(\sum_{i=u}^{k}\sum_{j=v}^{l}\frac{[X_{ij}]_\alpha^{(p)} - E[X_{ij}]_\alpha^{(p)}}{r_{ij}}\right).
\end{aligned}$$

Since $\Delta b_{mn} \geq 0$ for all $m \geq 1, n \geq 1$, then $\Delta r_{uv} \geq 0$ and we obtain

$$\begin{aligned}
&\max_{1\leq k\leq m, 1\leq l\leq n}\left|\frac{\sum_{i=1}^{k}\sum_{j=1}^{l}\left([X_{ij}]_\alpha^{(p)} - E[X_{ij}]_\alpha^{(p)}\right)}{r_{kl}}\right| \\
&\leq 4\max_{1\leq i\leq m, 1\leq j\leq n}\left|\sum_{u=1}^{i}\sum_{v=1}^{j}\frac{[X_{uv}]_\alpha^{(p)} - E[X_{uv}]_\alpha^{(p)}}{r_{uv}}\right|,
\end{aligned}$$

for each $p = 1, 2$. Therefore,

$$\begin{aligned}
&\max_{1\leq k\leq m, 1\leq l\leq n}\frac{1}{r_{kl}^2}D_*^2(S_{kl}, ES_{kl}) \\
&\leq \frac{1}{2}\int_0^1 \max_{1\leq k\leq m, 1\leq l\leq n}\left|\frac{\sum_{i=1}^{k}\sum_{j=1}^{l}\left([X_{ij}]_\alpha^{(1)} - E[X_{ij}]_\alpha^{(1)}\right)}{r_{kl}}\right|^2 d\alpha \\
&\quad + \frac{1}{2}\int_0^1 \max_{1\leq k\leq m, 1\leq l\leq n}\left|\frac{\sum_{i=1}^{k}\sum_{j=1}^{l}\left([X_{ij}]_\alpha^{(1)} - E[X_{ij}]_\alpha^{(1)}\right)}{r_{kl}}\right|^2 d\alpha \\
&\leq 8\int_0^1 \max_{1\leq i\leq m, 1\leq j\leq n}\left|\sum_{u=1}^{i}\sum_{v=1}^{j}\frac{[X_{uv}]_\alpha^{(1)} - E[X_{uv}]_\alpha^{(1)}}{r_{uv}}\right|^2 d\alpha \\
&\quad + 8\int_0^1 \max_{1\leq i\leq m, 1\leq j\leq n}\left|\sum_{u=1}^{i}\sum_{v=1}^{j}\frac{[X_{uv}]_\alpha^{(2)} - E[X_{uv}]_\alpha^{(2)}}{r_{uv}}\right|^2 d\alpha.
\end{aligned}$$

This implies that,

$$
\begin{aligned}
&\mathbf{P}\Big(\max_{s\le k\le m, t\le l\le n}\frac{1}{b_{kl}}D_*(S_{kl},ES_{kl})\ge\varepsilon\Big)\\
\le\;& \frac{32}{\varepsilon^2}\int\limits_{\Omega}\int\limits_0^1 \max_{1\le i\le m,1\le j\le n}\left|\sum_{u=1}^{i}\sum_{v=1}^{j}\frac{[X_{uv}]_\alpha^{(1)}-E[X_{uv}]_\alpha^{(1)}}{r_{uv}}\right|^2 d\alpha d\mathbf{P}\\
&+\frac{32}{\varepsilon^2}\int\limits_{\Omega}\int\limits_0^1 \max_{1\le i\le m,1\le j\le n}\left|\sum_{u=1}^{i}\sum_{v=1}^{j}\frac{[X_{uv}]_\alpha^{(2)}-E[X_{uv}]_\alpha^{(2)}}{r_{uv}}\right|^2 d\alpha d\mathbf{P}\\
=\;& \frac{32}{\varepsilon^2}\int\limits_0^1 E\max_{1\le i\le m,1\le j\le n}\left|\sum_{u=1}^{i}\sum_{v=1}^{j}\frac{[X_{uv}]_\alpha^{(1)}-E[X_{uv}]_\alpha^{(1)}}{r_{uv}}\right|^2 d\alpha\\
&+\frac{32}{\varepsilon^2}\int\limits_0^1 E\max_{1\le i\le m,1\le j\le n}\left|\sum_{u=1}^{i}\sum_{v=1}^{j}\frac{[X_{uv}]_\alpha^{(2)}-E[X_{uv}]_\alpha^{(2)}}{r_{uv}}\right|^2 d\alpha\\
&\qquad\text{(by Fubini's theorem).}
\end{aligned}
$$

Since $\{X_{mn}, m\ge 1, n\ge 1\}$ is a double array of level-wise negatively associated $\mathcal{U}$-valued random variables, then $\{[X_{mn}]_\alpha^{(1)}, m\ge 1, n\ge 1\}$ and $\{[X_{mn}]_\alpha^{(2)}, m\ge 1, n\ge 1\}$ are double arrays of negatively associated real-valued random variables for each $\alpha\in(0;1]$. Hence, for each $\alpha\in(0;1]$, $\{([X_{uv}]_\alpha^{(1)}-E[X_{uv}]_\alpha^{(1)})r_{uv}^{-1}, 1\le u\le i, 1\le v\le j\}$ and $\{([X_{uv}]_\alpha^{(2)}-E[X_{uv}]_\alpha^{(2)})r_{uv}^{-1}, 1\le u\le i, 1\le v\le j\}$ are arrays of negatively associated mean zero real-valued random variables. It follows from Proposition 3.4 that

$$
\begin{aligned}
&\mathbf{P}\Big(\max_{s\le k\le m, t\le l\le n}\frac{1}{b_{kl}}D_*(S_{kl},ES_{kl})\ge\varepsilon\Big)\\
\le\;& \frac{32C_2}{\varepsilon^2}\int\limits_0^1\sum_{i=1}^{m}\sum_{j=1}^{n}\left(E\left|\frac{[X_{ij}]_\alpha^{(1)}-E[X_{ij}]_\alpha^{(1)}}{r_{ij}}\right|^2+E\left|\frac{[X_{ij}]_\alpha^{(2)}-E[X_{ij}]_\alpha^{(2)}}{r_{ij}}\right|^2\right)d\alpha\\
=\;& \frac{64C_2}{\varepsilon^2}\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{1}{r_{ij}^2}E\int\limits_0^1\left(\frac{1}{2}\Big(\big([X_{ij}]_\alpha^{(1)}-E[X_{ij}]_\alpha^{(1)}\big)^2+\big([X_{ij}]_\alpha^{(2)}-E[X_{ij}]_\alpha^{(2)}\big)^2\Big)\right)d\alpha\\
=\;& \frac{C}{\varepsilon^2}\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{1}{r_{ij}^2}E\int\limits_0^1 d_*^2\big([X_{ij}]_\alpha,[EX_{ij}]_\alpha\big)d\alpha\\
=\;& \frac{C}{\varepsilon^2}\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{ED_*^2\big(X_{ij},EX_{ij}\big)}{r_{ij}^2}\\
=\;& \frac{C}{\varepsilon^2}\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{\mathrm{Var}X_{ij}}{r_{ij}^2}=\frac{C}{\varepsilon^2}\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{\mathrm{Var}X_{ij}}{(b_{ij}+b_{st})^2}.
\end{aligned}
$$

□

Theorem 3.6. *Let $\{b_{mn}, m \geq 1, n \geq 1\}$ be a 2-dimensional array of positive real numbers with $\Delta b_{mn} \geq 0$ for all $m \geq 1, n \geq 1$ and $\{X_{mn}, m \geq 1, n \geq 1\}$ be a double array of D_∞-integrable, level-wise negatively associated $\mathcal{U}$-valued random variables with $E\|X_{mn}\|_*^2 < \infty$, $m \geq 1, n \geq 1$. If*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\mathrm{Var}X_{mn}}{b_{mn}^2} < \infty, \tag{3.5}$$

then the strong law of large numbers

$$\frac{1}{b_{mn}}D_*(S_{mn}, ES_{mn}) \to 0 \text{ a.s. as } m \wedge n \to \infty \tag{3.6}$$

holds.

Proof. For $\varepsilon > 0$, by Theorem 3.5 we obtain

$$\begin{aligned}
&\mathbf{P}\Big(\sup_{k\geq s, l\geq t}\frac{1}{b_{kl}}D_*(S_{kl}, ES_{kl}) \geq \varepsilon\Big)\\
=\ & \lim_{m\wedge n\to\infty}\mathbf{P}\Big(\max_{s\leq k\leq m, t\leq l\leq n}\frac{1}{b_{kl}}D_*(S_{kl}, ES_{kl}) \geq \varepsilon\Big)\\
\leq\ & \lim_{m\wedge n\to\infty}\frac{C}{\varepsilon^2}\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{\mathrm{Var}X_{ij}}{(b_{ij}+b_{st})^2} = \frac{C}{\varepsilon^2}\sum_{i\geq 1}\sum_{j\geq 1}\frac{\mathrm{Var}X_{ij}}{(b_{ij}+b_{st})^2}\\
=\ & \frac{C}{\varepsilon^2}\Bigg\{\sum_{i=1}^{s}\sum_{j=1}^{t}\frac{\mathrm{Var}X_{ij}}{(b_{ij}+b_{st})^2} + \Big(\sum_{i\geq 1}\sum_{j\geq 1}\frac{\mathrm{Var}X_{ij}}{(b_{ij}+b_{st})^2} - \sum_{i=1}^{s}\sum_{j=1}^{t}\frac{\mathrm{Var}X_{ij}}{(b_{ij}+b_{st})^2}\Big)\Bigg\}\\
\leq\ & \frac{C}{\varepsilon^2}\frac{1}{b_{st}^2}\sum_{i=1}^{s}\sum_{j=1}^{t}\mathrm{Var}X_{ij} + \frac{C}{\varepsilon^2}\Big(\sum_{i\geq 1}\sum_{j\geq 1}\frac{\mathrm{Var}X_{ij}}{b_{ij}^2} - \sum_{i=1}^{s}\sum_{j=1}^{t}\frac{\mathrm{Var}X_{ij}}{b_{ij}^2}\Big).
\end{aligned}$$

By the condition (3.5) and Kronecker's lemma for multidimensional version (see Lemma 1.1 in [6]), we have

$$\frac{1}{b_{st}^2}\sum_{i=1}^{s}\sum_{j=1}^{t}\mathrm{Var}X_{ij} \to 0 \text{ as } s\wedge t \to \infty,$$

$$\text{and}\quad \Big(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{\mathrm{Var}X_{ij}}{b_{ij}^2} - \sum_{i=1}^{s}\sum_{j=1}^{t}\frac{\mathrm{Var}X_{ij}}{b_{ij}^2}\Big) \to 0 \text{ as } s\wedge t \to \infty.$$

Therefore, we get

$$\lim_{s\wedge t\to\infty}\mathbf{P}\Big(\sup_{k\geq s, l\geq t}\frac{1}{b_{kl}}D_*(S_{kl}, ES_{kl}) \geq \varepsilon\Big) = 0,$$

and this completes the proof. □

The following theorem establishes the weak law of large numbers for an array of level-wise negatively associated $\mathcal{U}$-valued random variables.

Theorem 3.7. *Let $\{X_{mn}, m \geq 1, n \geq 1\}$ be a double array of D_∞-integrable, level-wise negatively associated $\mathcal{U}$-valued random variables with $E\|X_{mn}\|_*^2 < \infty$, $m \geq 1, n \geq 1$. Assume that $\{b_{mn}, m \geq 1, n \geq 1\}$ be a 2-dimentional array of positive real numbers with $\Delta b_{mn} \geq 0$ for all $m \geq 1, n \geq 1$. If $b_{mn}^{-2} \sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} \mathrm{Var} X_{ij} \to 0$ as $m \vee n \to \infty$, then*

$$\frac{1}{b_{mn}} \max_{1\leq k\leq m, 1\leq l\leq n} D_*(S_{kl}, ES_{kl}) \to 0 \text{ in probability as } m \vee n \to \infty. \tag{3.7}$$

Proof. For any $\varepsilon > 0$, by Markov's inequality, we get

$$\mathbf{P}\Big(\frac{1}{b_{mn}} \max_{1\leq k\leq m, 1\leq l\leq n} D_*(S_{kl}, ES_{kl}) > \varepsilon\Big) \leq \frac{1}{\varepsilon^2 b_{mn}^2} E\Big(\max_{1\leq k\leq m, 1\leq l\leq n} D_*(S_{kl}, ES_{kl})\Big)^2.$$

Let us prove that

$$E\Big(\max_{1\leq k\leq m, 1\leq l\leq n} D_*(S_{kl}, ES_{kl})\Big)^2 \leq C \sum_{i=1}^{m} \sum_{j=1}^{n} \mathrm{Var} X_{ij},$$

where C is a positive constant which does not depend on m, n. Indeed, since $\{X_{mn}, m \geq 1, n \geq 1\}$ is a double array of level-wise NA $\mathcal{U}$-valued random variables, then $\{[X_{mn}]_\alpha^{(1)}, m \geq 1, n \geq 1\}$ and $\{[X_{mn}]_\alpha^{(2)}, m \geq 1, n \geq 1\}$ are arrays of negatively associated real-valued random variables for each $\alpha \in (0; 1]$. Hence

$$\begin{aligned}
&E\Big(\max_{1\leq k\leq m, 1\leq l\leq n} D_*(S_{kl}, ES_{kl})\Big)^2 = E \max_{1\leq k\leq m, 1\leq l\leq n} D_*^2(S_{kl}, ES_{kl}) \\
&= \int_\Omega \Big(\max_{1\leq k\leq m, 1\leq l\leq n} \int_0^1 d_*^2\big([S_{kl}]_\alpha, [ES_{kl}]_\alpha\big) d\alpha\Big) d\mathbf{P} \\
&\leq \int_\Omega \int_0^1 \max_{1\leq k\leq m, 1\leq l\leq n} d_*^2\big([S_{kl}]_\alpha, [ES_{kl}]_\alpha\big) d\alpha d\mathbf{P} \\
&= \int_0^1 E \max_{1\leq k\leq m, 1\leq l\leq n} d_*^2\big([S_{kl}]_\alpha, [ES_{kl}]_\alpha\big) d\alpha \text{ (by Fubini's theorem)} \\
&= \frac{1}{2} \int_0^1 E \max_{1\leq k\leq m, 1\leq l\leq n} \Big(\sum_{i=1}^{k} \sum_{j=1}^{l} \Big([X_{ij}]_\alpha^{(1)} - E[X_{ij}]_\alpha^{(1)}\Big)\Big)^2 d\alpha \\
&+ \frac{1}{2} \int_0^1 E \max_{1\leq k\leq m, 1\leq l\leq n} \Big(\sum_{i=1}^{k} \sum_{j=1}^{l} \Big([X_{ij}]_\alpha^{(2)} - E[X_{ij}]_\alpha^{(2)}\Big)\Big)^2 d\alpha \\
&\leq \frac{C}{2} \int_0^1 \sum_{i=1}^{m} \sum_{j=1}^{n} \Big(E\Big([X_{ij}]_\alpha^{(1)} - E[X_{ij}]_\alpha^{(1)}\Big)^2 + E\Big([X_{ij}]_\alpha^{(2)} - E[X_{ij}]_\alpha^{(2)}\Big)^2\Big) d\alpha
\end{aligned}$$

(by Proposition 3.4)

$$= C\sum_{i=1}^{m}\sum_{j=1}^{n}\int_0^1 E\Big(\frac{1}{2}\Big([X_{ij}]_\alpha^{(1)} - E[X_{ij}]_\alpha^{(1)}\Big)^2 + \frac{1}{2}\Big([X_{ij}]_\alpha^{(1)} - E[X_{ij}]_\alpha^{(1)}\Big)^2\Big)d\alpha.$$

Therefore, we obtain

$$\begin{aligned}
&E\Big(\max_{1\le k\le m, 1\le l\le n} D_*(S_{kl}, ES_{kl})\Big)^2\\
&= C\sum_{i=1}^{m}\sum_{j=1}^{n}E\int_0^1 \Big(\frac{1}{2}\Big([X_{ij}]_\alpha^{(1)} - E[X_{ij}]_\alpha^{(1)}\Big)^2 + \frac{1}{2}\Big([X_{ij}]_\alpha^{(1)} - E[X_{ij}]_\alpha^{(1)}\Big)^2\Big)d\alpha\\
&= C\sum_{i=1}^{m}\sum_{j=1}^{n}E\int_0^1 d_*^2([X_{ij}]_\alpha, [EX_{ij}]_\alpha)d\alpha = C\sum_{i=1}^{m}\sum_{j=1}^{n}ED_*^2(X_{ij}, EX_{ij})\\
&= C\sum_{i=1}^{m}\sum_{j=1}^{n}\mathrm{Var}X_{ij},
\end{aligned}$$

and this implies that the weak law of large numbers holds. □

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TÓM TẮT

LUẬT SỐ LỚN CHO MẢNG HAI CHIỀU CÁC HÀM NGẪU NHIÊN NỬA LIÊN TỤC TRÊN

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Ngày nhận bài 20/4/2021, ngày nhận đăng 30/6/2021

Trong bài báo này, chúng tôi giới thiệu một số luật số lớn cho mảng hai chiều các hàm ngẫu nhiên nửa liên tục trên liên kết âm theo mức với các giả thiết khác nhau. Chúng tôi cũng thiết lập một số bất đẳng thức cực đại cho cấu trúc hai chiều. Kết quả của chúng tôi là phần mở rộng cho các kết quả tương ứng trong các tài liệu.

Từ khóa: Biến ngẫu nhiên đa trị; liên kết âm theo mức; hàm ngẫu nhiên nửa liên tục trên.