AN ITERATED INFLATION FOR Q-BRAUER ALGEBRAS

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Abstract: In this article we construct an iterated inflation to the q-Brauer algebra. This construction enable us to show that the q-Brauer algebra is a cellular algebra.

Keywords: q-Brauer algebra; Algebra with iterated inflation; Cellular algebra.

1 Introduction

In 2012 a new algebra, called the q-Brauer algebra, was introduced by Wenzl [7] for works relating to the representation theory and C^*-algebra. Since q-Brauer algebra is a q-deformation of the classical Brauer algebra and contains the Hecke algebra of the symmetric group as a subalgebra, it is natural to ask whether this algebra is cellularly or not.

Cellular algebras have been stated by Graham and Lerher in [4]. In their construction an algebra is shown to possess cellular property if we can find a suitable cellular basis of this algebra. After that, using Graham and Lerher' approach, a large class of diagram algebras has been proven to be cellular algebras, such as Brauer algebra, group algebra of symmetric groups and its deformation, Iwahori Hecke algebra [4], Partition algebra [8], BMW-algebra [9]. In the case of the q-Brauer algebra, we do not know if this one is a diagram algebra. That is, it is unknown if there exists a basis represented by certain diagrams for the q-Brauer algebra. Instead of applying the method above, we are going to use Koenig and Xi's construction in [5] to give a suitable iterated inflation for the q-Brauer algebra. This iterated inflation enable us show the celluar structure for the q-Brauer algebra.

The article is organized as follows: In Section 2 we briefly review about the cellular algebras following [4, 5], and necessary facts on the classical Brauer algebra following [2] and [7]. Section 3 is used to construct an iterated inflation to the q-Brauer algebra.

2 Preliminaries

In this section we recall the original definition of cellular algebras in the sense of Graham and Lehrer in [4] and an equivalent definition given in [5] by Koenig and Xi.

2.1 Cellular algebras

(Graham and Lehrer [4]) Let R be a commutative Noetherian integral domain with identity. A *cellular algebra* over R is an associative (unital) algebra A together with *cell datum* (Λ, M, C, i), where

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(C1) Λ is a partially ordered set (poset) and for each $\lambda \in \Lambda$, $M(\lambda)$ is a finite set such that the algebra A has an R-basis $C^{\lambda}_{S,T}$, where (S,T) runs through all elements of $M(\lambda) \times M(\lambda)$ for all $\lambda \in \Lambda$.

(C2) Let $\lambda \in \Lambda$ and $S,T \in M(\lambda)$. Then i is an involution of A such that $i(C^{\lambda}_{S,T}) = C^{\lambda}_{T,S}$.

(C3) For each $\lambda \in \Lambda$ and $S,T \in M(\lambda)$ then for any element $a \in A$ we have

$$aC^{\lambda}_{S,T} \equiv \sum_{U \in M(\lambda)} r_a(U,S) C^{\lambda}_{U,T} \ (mod\ A(<\lambda)),$$

where $r_a(U,S) \in R$ is independent of T, and $A(<\lambda)$ is the R-submodule of A generated by $\{C^{\mu}_{S',T'} | \mu < \lambda;\ S^{'}, T^{'} \in M(\mu)\}$.

The basis $\{C^{\lambda}_{S,T}\}$ of a cellular algebra A is called a *cell basis*. In [4], Graham and Lehrer defined a bilinear form ϕ_λ for each $\lambda \in \Lambda$ with respect to this basis as follfows.

$$C^{\lambda}_{S,T} C^{\lambda}_{U,V} \equiv \phi_\lambda(T,U) C^{\lambda}_{S,V} \ (mod\ A < \lambda).$$

When R is a field, they also proved that the isomorphism classes of simple modules are parametrized by the set

$$\Lambda_0 = \{\lambda \in \Lambda |\ \phi_\lambda \neq 0\}.$$

The following is an equivalent definition of cellular algebra.

(Koenig and Xi [5]) Let A be an R-algebra where R is a commutative noetherian integral domain. Assume there is an involution i on A. A two-sided ideal J in A is called *cell ideal* if and only if$i(J) = J$ and there exists a left ideal $\Delta \subset J$ such that Δ is finitely generated and free over R and such that there is an isomorphism of A-bimodules $\alpha\ :\ J \simeq \Delta \otimes_R i(\Delta)$ (where $i(\Delta) \subset J$ is the i-image of Δ) making the following diagram commutative:

$$\begin{array}{ccc} J & \xrightarrow{\alpha} & \Delta \otimes_R i(\Delta) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle x\otimes y \mapsto i(y)\otimes i(x)} \\ J & \xrightarrow{\alpha} & \Delta \otimes_R i(\Delta) \end{array}$$

The algebra A with the involution i is called cellular $if and only if$ there is an R-module decomposition $A = J^{'}_1 \oplus J^{'}_2 \oplus ... J^{'}_n$ (for some n) with $i(J^{'}_j) = J^{'}_j$ for each j and such that setting $J_j = \oplus^{j}_{l=1} J^{'}_l$ gives a chain of two-sided ideals of A: $0 = J_0 \subset J_1 \subset J_2 \subset ... \subset J_n = A$ (each of them fixed by i) and for each j $(j = 1, ..., n)$ the quotient $J^{'}_j = J_j / J_{j-1}$ is a cell ideal (with respect to the involution induced by i on the quotient) of A/J_{j-1}.

Recall that an involution i is defined as an R-linear anti-automorphism of A with $i^2 = id$. The $\Delta^{'}s$ obtained from each section J_j/J_{j-1} are called *cell modules* of the cellular algebra A. Note that all simple modules are obtained from cell modules [4].

In [5], Koenig and Xi proved that the two definitions of cellular algebra are equivalent. The first definition can be used to check concrete examples, the latter, however, is convenient to look at the structure of cellular algebras as well as to check cellularity of an algebra.The difference between these two approachs is that in [4] the cellular property of an algebra is shown by finding a suitable cellular basis, but in the other way we only construct a good iterated inflation to the algebra. If Graham and Lerher's method needs more knowlegde in combinatorics area, then Koenig and Xi's approach uses language of ring and module theory.

In the later koenig and Xi showed that any cellular algebra can be exhibited as an iterated inflation of copies of the ground ring R, and conversely, any iterated inflation of cellular algebras is also cellular. Here, we represent briefly how an iterated inflation algebra is formed.

Step 1. Inflating algebras along free R-modules

Let B be cellular R-algebra, V a finitely generated free R-module, and a bilinear form

$$\varphi : V \otimes_R V \longrightarrow B, \tag{2.1}$$

with values in B. An associative algebra (possibly without unit) $A(B, V, \varphi)$ is defined as follows : as an R-module, A equals $V \otimes_R V \longrightarrow B$. The multiplication on A is defined on basis elements:

$$(a \otimes b \otimes x) \cdot (c \otimes d \otimes y) := (a \otimes d \otimes x\varphi(b, c)y.$$

If assume more that j is an involution of B and φ satisfies $j(\varphi(v, w)) = \varphi(w, v)$. Then an involution i on A is set up by putting

$$i(a \otimes b \otimes x) = b \otimes a \otimes j(x).$$

This construction makes A into an associative R-algebra with an anti-automorphism i, and such algebra A is called an *inflation* of B along V. The algebra A need not have a unit element, but it may have idempotent elements.

Step 2. Inflating an algebra along another one

Let B be an algebra (possibly without unit) and C an algebra (with unit). This step aims to define an extended algebra structure on $A := B \oplus C$ such that B becomes a two-sided ideal and A/B isomorphic to C. Such construction requires conditions to make sure that the multiplication is associative and the unit element of A is mapped to the unit of C by the quotient homomorphism. The paticular description of these conditions is outlined in [5].

Now, let C be any algebra (with unit) and let B be an algebra of the form $V \otimes_R V \longrightarrow B'$ as step 1 of the construction. Let $A := B \oplus C$ be as in step 2. We call A an inflation of C along B. The result of iterated application of this construction is called an *iterated inflation.* It is well-known that a cellular algebra may have different iterated inflations.

To show the main result we need the following lemma. This lemma is shown in [9] as a condition to ensure that an algebra has cellular structure.

Lemma 2.1. *[8], Lemma 3.3) Let A be a $\Lambda-$algebra with an involution i. Suppose there is a decomposition*

$$A = \oplus_{j=1}^{m} V_{(j)} \otimes_\Lambda V_{(j)} \otimes_\Lambda B_j \quad (\textit{direct sum of } \Lambda-\textit{modules})$$

where $V_{(j)}$ is a free $\Lambda-$module of finite rank and B_j is a cellular $\Lambda-$algebra with respect to an involution δ_j and a cell chain $J_1^j \subset J_2^j ... \subset J_{s_j}^j = B_j$ for each j. Define $J_t = \oplus_{j=1}^t V_{(j)} \otimes_\Lambda V_{(j)} \otimes_\Lambda B_j$. Assume that the restriction of i on $\oplus_{j=1}^t V_{(j)} \otimes_\Lambda V_{(j)} \otimes_\Lambda B_j$ is given by $w \otimes v \otimes b \longrightarrow v \otimes w \otimes \delta_j(b)$. If for each j there is a bilinear form $\varphi_j : V_{(j)} \otimes_\Lambda V_{(j)} \longrightarrow B_j$ such that $\delta_j(\varphi_j(w,v)) = \varphi_j(v,w)$ for all $w, v \in V_{(j)}$ and that the multiplication of two elements in $V_{(j)} \otimes_\Lambda V_{(j)} \otimes_\Lambda B_j$ is governed by ϕ_j modulo J_{j-1}; that is, for $x, y, u, v \in V_{(j)}$, and $b, c \in B_j$, we have

$$(x \otimes y \otimes b)(u \otimes v \otimes c) = x \otimes v \otimes b\varphi_j(y,u)c$$

modulo the ideal J_{j-1}, and if $V_{(j)} \otimes_\Lambda V_{(j)} \otimes_\Lambda J_l^j + J_{j-1}$ is an ideal in A for all l and j, then A is a cellular algebra.

2.2 Brauer algebras

In 1937 Brauer [1] introduced an algebra to play a role in Schur-Weyl duality when replacing a general linear group $GL(^N)$ by its groups, a symplectic or an orthogonal group, and in the other side the symmetric group is substituted by his algebra. The Brauer algebra has a basis of diagrams, each a diagram consists of $2n$ dots arranged in two rows and n edges, where each dot belongs to exactly one edge. The edges which connect two dots on the same row are said *horizontal edges*. The other ones are called *vertical edges*. The multiplication of two diagrams h_1 and h_2 is a concatenation in the following way: put diagram h_1 on top of d_2 such that all dots in the bottom row of h_1 are straight column with all upper dots of h_2. Now draw an edge from dot i in the bottom row of h_1 to dot i in top row of h_2 for all i. The resulting diagram, consists of parts that start and finish in top row of h_1 and bottom row of h_2 respectively, as well as some cycles that use only dots in the middle two rows. The product $h_1 \cdot h_2$ is then defined to be this resulting diagram without internal cycles, multiplied by x taken to the power of the number of internal cycles appeared above. Here x is a variable. Denote $D_n(x)$ to be the Brauer algebra consisting of $2n$ dots over a ground ring. For more details about the Brauer algebra we refer reader to, for example, [1, 4-6].

It is mentioned in [1] that the Brauer algebra possess set of generators s_i, $1 \le i \le n-1$. Let k be an integer, $0 \le k \le [n/2]$. Define J_k to be the F-vector space with basis consisting of all diagrams with at least $2k$ horizontal edges. That is, there are at least k edges in a row of the diagram and each of them joins exactly two dots in the row. Then J_k is a two-sided ideal of $D_n(N)$ and it implies a filtration of the Brauer algebra:

$$\{0\} \subsetneq J_{[n/2]} \subseteq J_{[n/2]-1} \subseteq \dots \; J_0 = D_n(N).$$

where $J_{[n/2]} = \{0\}$ or $J_{[n/2]} \ne \{0\}$ depends on whether n is even or odd. The subquotient J_k/J_{k+1} is isomorphic to an inflation $V_{n-2k} \otimes_F V_{n-2k} \otimes_F \Sigma_{n-2k}$ of Σ_{n-2k} along a vector space V_{n-2k} as given in Lemma 5.3 [6]. As a consequence, Lemma 2.1 yields that J_k/J_{k+1} is also isomorphic to an inflation $\bar{U}_k^* \otimes_F \bar{U}_k \otimes_F \Sigma_{2k+1,n}$. This realizes $D_n(N)$ as an iterated inflation of group algebras of symmetric groups, which can be stated similarly as in Theorem 5.6 [6].

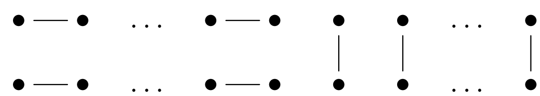
Length function for Brauer algebras

This section will be used in Section 3. Throughout, let F be a field of any characteristic, and fix $x = N$ with $N \in F$. For more details we refer the reader to [2], Section 2 (or [7], Section 1.4).

Generalizing the length of elements in reflection groups, Wenzl [7] defined a length function for a diagram of $D_n(N)$ as follows: Let k be an integer, $0 \leq k \leq [n/2]$. For a diagram $d \in D_n(N)$ with exactly $2k$ horizontal edges, the definition of the length $\ell(d)$ is given by

$$\ell(d) = min\{\ell(w_1) + \ell(w_2)| \quad w_1 e_{(k)} w_2 = d,\ w_1, w_2 \in \Sigma_n\},$$

where $e_{(k)}$ to be the following diagram:

$e_{(k)} =$  (2.2)

where each row has exactly k horizontal edges.

Recall for a permutation $w \in \Sigma_n$ an expression $w = s_{i_1} s_{i_2} \cdots s_{i_m}$ in which m is minimal is called a *reduced expression* for w, and $\ell(w) = m$ is called the length of a permutation w. Notice that a permutation $w\Sigma_n$ can be seen as a diagram with no horizontal edge, and in this case the length function in $D_n(N)$ above restricts to this in Σ_n.

A permutation $w \in \Sigma_n$ can be written uniquely in the form $w = t_1 \dots t_{n-2} t_{n-1}$, where $t_j = 1$ or $t_j = s_{j,i_j} := s_j s_{j-1} \cdots s_{i_j}$ with $1 \leq i_j \leq j < n$.

Denote

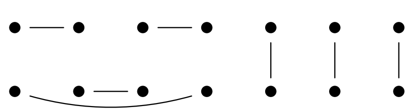
$$\mathfrak{B}_{k,n} = \{w \in \Sigma_n |\ w = t_2 t_4 \dots t_{2k-2} t_{2k} t_{2k+1} \dots t_{n-2} t_{n-1}\}.$$

By the definition of t_j given above, the number of possibilities of t_j is $j+1$. A direct computation shows that $\mathfrak{B}_{k,n}$ has $n!/2^k k!$ elements. In fact, the number of elements in $\mathfrak{B}_{k,n}$ is equal to the number of diagrams d in $D_n(N)$ in which d has exactly $2k$ horizontal edges and its top row is fixed like that of $e_{(k)}$.

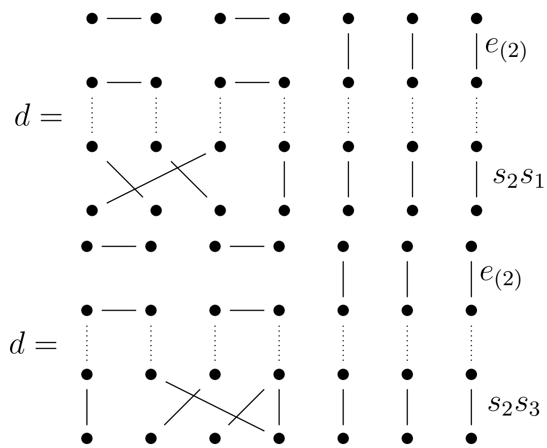
Given a diagram d which has exactly $2k$ horizontal edges and its top row is fixed like that of $e_{(k)}$, there exist different diagrams $w_1, w_2 \in \Sigma_n$ such that $d = e_{(k)} w_1 = e_{(k)} w_2$ and $\ell(d) = \ell(w_1) = \ell(w_2)$. However, it is implicit in [7], Lemma 3.2 that there exists either w_1 or w_2 in $\mathfrak{B}_{k,n}$.

Let us illustrate this by the following example.

In the Brauer algebra $D_7(N)$, d is the below diagram

$d =$ 

Then, d has two representations as follows:



It is obviously that $d = e_{(2)}s_2s_1 = e_{(2)}s_2s_3$ satisfying $\ell(d^*) = \ell(s_2s_1) = \ell(s_2s_3) = 2$. However, $t_2 = s_2s_1$ is in $\mathfrak{B}_{2,7}$ but $t_2t_3 = s_2s_3$ is not.

Let $\mathscr{D}_k$ be a set of all diagrams d which such a diagram has the top row like that of the diagram $e_{(k)}$ and no intersection between any two vertical edges. Similarly, let $\mathscr{D}_k^*$ be a set of all diagrams d^* which are obtained by reflecting diagrams $d \in \mathscr{D}_k$ via a horizontal axis. Denote

$$\mathfrak{B}_k = \{w \in \mathfrak{B}_{k,n} | \text{ such that } d \in \mathscr{D}_k, d = e_{(k)}w, \text{ and } \ell(d) = \ell(w)\}$$

and

$$\mathfrak{B}_k^* = \{w^{-1} | \text{ with } w \in \mathfrak{B}_k\}.$$

It is clear to check straightforward that

$$|\mathfrak{B}_k| = |\mathfrak{B}_k^*| = dim_F F\mathscr{D}_k = dim_F F\mathscr{D}_k^* = n!/2^k(n-2k)!k!. \quad (2.3)$$

The following statement is another version of Lemma 4.1 in [2].

Lemma 2.2. *The F-vector space $F\mathscr{D}_k$ has a basis $\{d := e_{(k)}w$ with $w \in \mathfrak{B}_k\}$. Dually, the F-vector space $F\mathscr{D}_k^*$ has a basis $\{d^* := w^{-1}e_{(k)}$ with $w \in \mathfrak{B}_k\}$.*

A reduced expression of a diagram

Let us recall briefly the concept *"reduced expression"* of a diagram introduced in [2].

Let h be an arbitrary diagram in $D_n(N)$ with exactly $2k$ horizontal edges. It always can be uniquely represented as concatenation of three partial diagrams d^*, d and ω, where $d \in \mathscr{D}_k$, $d^* \in \mathscr{D}_k^*$, and $\omega \in \Sigma_{2k+1,n}$ analyzed in [2], Section 3.2. Then, by Lemma 2.2 there exist unique elements $w_1^{-1}, w_2 \in \mathfrak{B}_k$ such that $d = e_{(k)}w_2$ and $d^* = w_1e_{(k)}$ with $\ell(d) = \ell(w_2)$ and $\ell(d) = \ell(w_1)$. This means h is uniquely represented by the triple (w_1, w_2, ω) with $\ell(h) = \ell(w_1) + \ell(\omega) + \ell(w_2)$, where $w_1^{-1}, w_2 \in \mathfrak{B}_k$ and $\omega \in \Sigma_{2k+1,n}$. Such a triple (w_1, w_2, ω) is called a *reduced expression* of the diagram h.

In Example 2.2, the diagram d has a reduced expression $(\mathbf{1}_{D_7(N)}, s_2s_1, \mathbf{1}_{\Sigma_{5,7}})$. Another expression of d is $(\mathbf{1}_{D_7(N)}, s_2s_3, \mathbf{1}_{\Sigma_{5,5}})$, but this does not fit into our above definition.

3 An iterarted inflation for q-Brauer algebras

3.1 The q-Brauer algebras

The q-Brauer algebra is a deformation of the Brauer algebra, which is introduced by Wenzl [7]. This algebra contains the Hecke algebra of symmetric group as a natural subalgebra. Let

$$[N] = 1 + q^1 + \cdots + q^{N-1} \in [q, q^{-1}],\ n \in,\ \text{and}\ N \in \backslash\{0\}.$$

The q-Brauer algebra $Br_n(N)$ is defined over ring $[q, q^{-1}]$ by generators g_1, g_2, g_3, ..., g_{n-1} and e and relations

(H) The elements g_1, g_2, g_3, ..., g_{n-1} satisfy the relations of the Hecke algebra $H_n(q)$. That is: $g_ig_j = g_jg_i$ for $|i-j| > 1$, $g_ig_{i+1}g_i = g_{i+1}g_ig_{i+1}$ for $1 \leq i \leq n-2$, and $g_i^2 = (q-1)g_i + q$ for $1 \leq i \leq n-1$;

$(E1)$ $e^2 = [N]e$;

$(E2)$ $eg_i = g_ie$ for $i > 2$, $eg_1 = g_1e = qe$, $eg_2e = q^Ne$ and $eg_2^{-1}e = q^{-1}e$;

(E_3) $e_{(2)} = g_2g_3g_1^{-1}g_2^{-1}e_{(2)} = e_{(2)}g_2g_3g_1^{-1}g_2^{-1}$, where $e_{(2)} = e(g_2g_3g_1^{-1}g_2^{-1})e$.

The elements $e_{(k)}$ in $Br_n(N)$ are defined inductively by $e_{(1)} = e$ and by

$$e_{(k+1)} = eg^+_{2,2k+1}g^-_{1,2k}e_{(k)}. \tag{3.1}$$

Let us briefly review some basic facts of the q-Brauer algebra shown in [2]. It possesses an explicit basis $\{g_h\}$ labeled by diagrams h of the Brauer algebra. In particular, give a diagram h with a reduced expression (w_1, w_2, ω), where $w_1^{-1}, w_2 \in \mathfrak{B}_k$ and $\omega \in \Sigma_{2k+1,n}$. Then, the set

$$\{g_h := g_{w_1}e_{(k)}g_\omega g_{w_2} |\ \ h \text{ is a diagram in } D_n(N)\} \tag{3.2}$$

is a basis of the q-Brauer algebra. Define J_k a $[q, q^{-1}]$-module generated by elements g_h where h are diagrams with at least $2k$ horizontal edges. This definition yields J_k is two-sided ideal in $Br_n(N)$ and

$$J_k = \sum_{j=k}^{[n/2]} H_n(q)e_{(j)}H_n(q). \tag{3.3}$$

Hence,

$$\{0\} \subsetneq J_{[n/2]} \subseteq J_{[n/2]-1} \subseteq \ldots\ J_0 = Br_n(N)$$

is a chain of two-sided ideals. The involution i on $Br_n(N)$ is defined by the rule $i(e) = e$, $i(g_w) = g_{w^{-1}}$ for $w \in \Sigma_n$. It is clearly that i is compatible with the usual involution of the Hecke algebra.

The below properties of the q-Brauer algebra $Br_n(N)$ are collected from Lemma 3.4 [7] and Corollary 3.15 [2]. These will be used in the following section.

Lemma 3.1. *Let k, l are integers, $0 \leq k, l \leq [n/2]$. Then*

1. $e_{(k)}e_{(l)} = e_{(l)}e_{(k)} = [N]^l e_{(k)}$ *for* $k \geq l$.
2. $e_{(k)}H_n e_{(l)} \subset H_{2k+1,n}e_{(l)} + \sum_{m \geq l+1} H_n e_{(m)} H_n$ *if* $l \geq k$.
3. $e_{(k)}H_n e_{(l)} \subset e_{(k)}H_{2l+1,n} + \sum_{m \geq k+1} H_n e_{(m)} H_n$ *if* $k \geq l$.

3.2 The construction of an iterated inflation

This section is devoted to give a construction for the main Theorem. We are going to apply two steps in Section 2.1 to set up an iterated inflation for the q-Brauer algebra.

Throughout, let F be an arbitrary field of any characteristic. Assume more that q, $[N]$ are invertible in F.

For k an integer, $0 \leq k \leq [n/2]$, denote $\bar{e}_{(k)} := \frac{1}{[N]^k} e_{(k)}$.

Define U_k to be a F-vector space spanned by the set

$$\{\frac{1}{[N]^k} g_d \mid \ d \text{ is a diagram in } \mathscr{D}_k\}$$

and let $U_k^* := R\{\frac{1}{[N]^k} g_{d^*} \mid d^* \text{ is a diagram in } \mathscr{D}_k^*\}$.

Lemma 3.2. *The F-module U_k has a basis $\{g_\omega \bar{e}_{(k)} \mid \omega \in \mathfrak{B}_k\}$. Dually, the F-module U_k^* has a basis $\{\bar{e}_{(k)} g_\omega^{-1} \mid \omega \in \mathfrak{B}_k\}$. Moreover,*

$$dim_F U_k = dim_F U_k^* = \frac{n!}{2^k (n-2k)! k!}.$$

Proof. For a diagram $d \in \mathscr{D}_k$, by Lemma 2.2 and Definition (3.2) g_d is a basis element of the q-Brauer algebra and is in the form $g_d = e_{(k)} g_\omega$ where $\omega \in \mathfrak{B}_k$. It is obviously that the elements $\frac{1}{[N]^k} g_d$ $(= \frac{1}{[N]^k} g_\omega e_{(k)})$ with $\omega \in \mathfrak{B}_k$ are independent. The proof is the same for U_k^*. Thus, we get the precise statement. □

Proposition 3.3. *Fix an index k and let $B := J_k / J_{k+1}$ be the F-algebra (possibly without identity). Then B is isomorphic (as F-algebra) to an inflation $U_k^* \otimes_F U_k \otimes_F H_{2k+1,n}(q)$ of the Hecke algebra of the symmetric group $H_{2k+1,n}(q)$ along free F-modules U_k, U_k^*. The F-bilinear form*

$$\phi : U_k \otimes_F U_k^* \longrightarrow H_{2k+1,n}(q)$$

is determined by

$$\phi(\bar{e}_{(k)} g_v, g_u \bar{e}_{(k)}) \bar{e}_{(k)} := \bar{e}_{(k)} g_v \cdot g_u \bar{e}_{(k)} \mod J_{k+1},$$

where u^{-1}, v are in $\mathfrak{B}_k$.

Proof. Obviously, by (3.3) it implies that $B = J_k/J_{k+1} \cong H_n(q)(q)e_{(k)}H_n(q)(q)$ as F-vector spaces. Hence, $dim_F B = dim_F H_n(q)e_{(k)}H_n(q)$. Using Theorem 3.8(b) in [7], $dim_F H_n(q)e_{(k)}H_n(q)$ is equal to the number of all diagrams d which has exactly $2k$ horizontal edges. By direct calculation we obtain

$$dim_F H_n(q)e_{(k)}H_n(q) = dim_F B = dim_F U_k^* \otimes_F U_k \otimes_F H_{2k+1,n}(q) = \frac{(n!)^2}{4^k(n-2k)!(k!)^2}.$$

By the basis in (3.2), B is a vector space with F-basis

$$\{g_d := g_{w_1}e_{(k)}g_\omega g_{w_2} \mid d \text{ a diagram with exactly } 2k \text{ horizontal edges in } D_n(N)$$
$$(w_1, w_2, \omega) \text{is a \textit{reduced expression} of the diagram } d\}.$$

Notice that B is an F- associative algebra (without identity) and the multiplication on B is generated by that of the q-Brauer algebra modulo with J_{k+1}.
Denote $f_k : B \longrightarrow U_k^* \otimes_F U_k \otimes_F H_{2k+1,n}(q)$ to be a map determined by

$$g_{w_1}e_{(k)}g_\omega g_{w_2} \mapsto [N]^k(g_{w_1}\bar{e}_{(k)} \otimes \bar{e}_{(k)}g_{w_2} \otimes g_\omega),$$

where (w_1, ω, w_2) is a reduced expression of d.

This definition provides an isomorphism of F-modules. In order to show f_k is an algebra isomorphism, we need to define a multiplication on $U_k^* \otimes_F U_k \otimes_F H_{2k+1,n}(q)$ using the F-bilinear form ϕ as follows:

For $u_1^{-1}, u_2^{-1}, v_1, v_2 \in \mathfrak{B}_k$ and $\omega_1, \omega_2 \in \Sigma_{2k+1,n}$, define

$$\begin{aligned}(g_{u_1}\bar{e}_{(k)} \otimes \bar{e}_{(k)}g_{v_1} \otimes g_{\omega_1})\cdot(g_{u_2}\bar{e}_{(k)} \otimes \bar{e}_{(k)}g_{v_2} \otimes g_{\omega_2}) \qquad (3.4)\\ := g_{u_1}\bar{e}_{(k)} \otimes \bar{e}_{(k)}g_{v_2} \otimes g_{\omega_1}\phi(\bar{e}_{(k)}g_{v_1}, g_{u_2}\bar{e}_{(k)})g_{\omega_2}.\end{aligned}$$

By Theorem 3.1 in [5] the multiplication makes $U_k^* \otimes_F U_k \otimes_F H_{2k+1,n}(q)$ into an associative algebra (possibly without identity). It is left to verify that f_k is a ring isomorphism. To this end, pick up two arbitrary basis elements $g_{d_1}, g_{d_2} \in B$ up and assume that $g_{d_1} = g_{u_1}e_{(k)}g_{\omega_1}g_{v_1}$, $g_{d_2} = g_{u_2}e_{(k)}g_{\omega_2}g_{v_2}$. Then, we obtain

$$\begin{aligned} g_{d_1} \cdot g_{d_2} \bmod J_{k+1} &\equiv (g_{u_1}e_{(k)}g_{\omega_1}g_{v_1})(g_{u_2}e_{(k)}g_{\omega_2}g_{v_2}) \bmod J_{k+1} \qquad (3.5)\\ &\overset{L3.1(1)}{\equiv} (g_{u_1}e_{(k)}g_{\omega_1})(\bar{e}_{(k)}g_{v_1}g_{u_2}\bar{e}_{(k)})(e_{(k)}g_{\omega_2}g_{v_2}) \bmod J_{k+1}\\ &\overset{\phi}{=} (g_{u_1}e_{(k)}g_{\omega_1})(\phi(\bar{e}_{(k)}g_{v_1}, g_{u_2}\bar{e}_{(k)})\bar{e}_{(k)})(e_{(k)}g_{\omega_2}g_{v_2})\\ &\overset{L3.1(1)}{=} (g_{u_1}e_{(k)}g_{\omega_1})(\phi(\bar{e}_{(k)}g_{v_1}, g_{u_2}\bar{e}_{(k)}))(e_{(k)}g_{\omega_2}g_{v_2})\\ &= (g_{u_1}e_{(k)})(g_{\omega_1}\phi(\bar{e}_{(k)}g_{v_1}, g_{u_2}\bar{e}_{(k)})g_{\omega_2})(e_{(k)}g_{v_2}).\end{aligned}$$

Since $g_{\omega_1}\phi(\bar{e}_{(k)}g_{v_1}, g_{u_2}\bar{e}_{(k)})g_{\omega_2} \in H_{2k+1,n}(q)$, it can be represented as an F-linear combination of basis elements g_w with $w \in \Sigma_{2k+1,n}$. That is,

$$g_{\omega_1}\phi(\bar{e}_{(k)}g_{v_1} \otimes g_{u_2}\bar{e}_{(k)})g_{\omega_2} = \sum_{w \in \Sigma_{2k+1,n}} a_w g_w, \qquad (3.6)$$

where a_w are coefficients in F. Putting this formula into the equation (3.5), it yields

$$\begin{aligned} g_{h_1}\cdot g_{h_2} &\overset{(3.6)}{=} (g_{u_1}e_{(k)})(\sum_{w\in\Sigma_{2k+1,n}} a_w g_w)(e_{(k)}g_{v_2}) \\ &= \sum_{w\in\Sigma_{2k+1,n}} a_w(g_{u_1}e_{(k)}g_w e_{(k)}g_{v_2}). \end{aligned} \tag{3.7}$$

Hence,

$$\begin{aligned} f_k(g_{h_1}\cdot g_{h_2}) &\overset{(3.7)}{=} f_k(\sum_{w\in\Sigma_{2k+1,n}} a_w(g_{u_1}e_{(k)}g_w e_{(k)}g_{v_2})) \\ &= \sum_{w\in\Sigma_{2k+1,n}} a_w f_k(g_{u_1}e_{(k)}g_w e_{(k)}g_{v_2}) \\ &\overset{L3.1(1)}{=} \sum_{w\in\Sigma_{2k+1,n}} [N]^k a_w f_k(g_{u_1}e_{(k)}g_w g_{v_2}) \\ &\overset{f_k}{=} \sum_{w\in\Sigma_{2k+1,n}} [N]^{2k} a_w(g_{u_1}\bar{e}_{(k)}\otimes\bar{e}_{(k)}g_{v_2}\otimes g_w). \end{aligned}$$

In other words, we also have

$$\begin{aligned} f_k(g_{h_1})\cdot f_k(g_{h_2}) &\overset{f_k}{=} [N]^k(g_{u_1}\bar{e}_{(k)}\otimes\bar{e}_{(k)}g_{v_1}\otimes g_{\omega_1})\cdot[N]^k(g_{u_2}\bar{e}_{(k)}\otimes\bar{e}_{(k)}g_{v_2}\otimes g_{\omega_2}) \\ &\overset{(3.4)}{=} [N]^{2k}g_{u_1}\bar{e}_{(k)}\otimes\bar{e}_{(k)}g_{v_2}\otimes g_{\omega_1}\phi(\bar{e}_{(k)}g_{v_1}, g_{u_2}\bar{e}_{(k)})g_{\omega_2} \\ &\overset{(3.6)}{=} [N]^{2k}(g_{u_1}\bar{e}_{(k)}\otimes\bar{e}_{(k)}g_{v_2}\otimes\sum_{w\in\Sigma_{2k+1,n}} a_w g_w) \\ &= \sum_{w\in\Sigma_{2k+1,n}} [N]^{2k}a_w(g_{u_1}\bar{e}_{(k)}\otimes\bar{e}_{(k)}g_{v_2}\otimes g_w). \end{aligned}$$

Hence, $f_k(g_{h_1}\cdot g_{h_2}) = f_k(g_{h_1})\cdot f_k(g_{h_2})$. Thus, f_k is an algebra isomorphism. □

The next statement can be verified directly.

Lemma 3.4. *Under f_k the involution $i: B\to B$ corresponds to the involution on $U_k^*\otimes_F U_k\otimes_F H_{2k+1,n}(q)$ which sends $g_u\bar{e}_{(k)}\otimes\bar{e}_{(k)}g_v\otimes g_\omega$ to $g_{v^{-1}}\bar{e}_{(k)}\otimes\bar{e}_{(k)}g_{u^{-1}}\otimes g_{\omega^{-1}}$, where $u^{-1}, v\in\mathfrak{B}_k$ and $\omega\in\Sigma_{2k+1,n}$.*

Recall that i is the involution in the q-Brauer algebra determined by $i(e_{(k)}) = e_{(k)}$ and $i(g_\omega) = g_\omega^{-1}$ with $\omega\in\Sigma_n$.

We next show that the layers fit together (which is more than just having a filtration by two-sided ideals).

For k, l non-negative integers, $0\le k, l\le [n/2]$, let $x\in U_k^*\otimes_F U_k\otimes_R H_{2k+1,n}(q)$ and $y\in U_l^*\otimes_R U_l\otimes_R H_{2l+1,n}(q)$. Define

$$x\cdot y := f_k(f_k^{-1}(x)\cdot f_l^{-1}(y)\ mod\ J_{k+1}),$$

where f_j, $0 \leq j \leq [n/2]$, is the isomorphism defined in Proposition 3.3. Note that the product in the right hand-side of definition is the usual multiplication in the q-Brauer algebra. When $k = l$, the definition above recovers Definition (3.4). This claim is a consequence of the following lemma.

Lemma 3.5. *For k, l non-negative integers, $0 \leq k, l \leq [n/2]$, let $g_{h_1} := g_{u_1} e_{(k)} g_{\omega_1} g_{v_1}$ in $J_k \setminus J_{k+1}$ and $g_{h_2} := g_{u_2} e_{(l)} g_{\omega_2} g_{v_2}$ in $J_l \setminus J_{l+1}$ be two basis elements in the q-Brauer algebra, and let their respective f-images be $g_{u_1} \bar{e}_{(k)} \otimes \bar{e}_{(k)} g_{v_1} \otimes g_{\omega_1}$ and $g_{u_2} \bar{e}_{(l)} \otimes \bar{e}_{(l)} g_{v_2} \otimes g_{\omega_2}$. Then the product $g_{h_1} \cdot g_{h_2}$ either is an element of J_{k+1} or is an element of $J_k \setminus J_{k+1}$, and in the latter case it corresponds under f to a scalar multiple of an element $g_{u_1} \bar{e}_{(k)} \otimes b \otimes g_{\omega_1} c$ where b is an element in U_k and c is an element in $H_{2k+1,n}(q)$*

Proof. We separately consider two cases of l and k.
Case 1. If $l > k$, then Lemma 3.1(2) implies that

$$e_{(k)} g_{v_1} \cdot g_{u_2} g_{\omega_2} e_{(l)} \in H_{2k+1,n}(q) e_{(l)} + \sum_{m \geq l+1} H_n e_{(m)} H_n,$$

and hence

$$\begin{aligned}(g_{u_1} e_{(k)} g_{\omega_1} g_{v_1}) \cdot (g_{u_2} e_{(l)} g_{\omega_2} g_{v_2}) &= g_{u_1} g_{\omega_1} (e_{(k)} g_{v_1} \cdot g_{u_2} g_{\omega_2} e_{(l)}) g_{v_2} \\ &\in g_{u_1} g_{\omega_1} H_{2k+1,n}(q) e_{(l)} g_{v_2} + \sum_{m \geq l+1} H_n e_{(m)} H_n \\ &\subseteq H_n e_{(l)} H_n + \sum_{m \geq l+1} H_n e_{(m)} H_n \\ &= \sum_{m \geq l} H_n e_{(m)} H_n \overset{(??)}{=} J_l \overset{l>k}{\subseteq} J_{k+1}.\end{aligned}$$

Thus, $g_{h_1} \cdot g_{h_2} \equiv 0$ $(mod\ J_{k+1})$, that is, $g_{h_1} \cdot g_{h_2} \in J_k$.

Case 2. If $l \leq k$, then by Lemma 3.1(2), we obtain

$$e_{(k)} g_{v_1} \cdot g_{u_2} e_{(l)} \in e_{(k)} H_{2l+1,n} + \sum_{m \geq k+1} H_n e_{(m)} H_n.$$

Hence,

$$e_{(k)} g_{v_1} \cdot g_{u_2} e_{(l)} g_{\omega_2} g_{v_2} \in e_{(k)} H_{2l+1,n} g_{v_2} + \sum_{m \geq k+1} H_n e_{(m)} H_n.$$

Since $v_2 \in \mathfrak{B}_l$, using a dual statement of Lemma 4.10 [2] the product $e_{(k)} g_{v_1} \cdot g_{u_2} g_{\omega_2} e_{(l)} g_{v_2}$ can be rewritten as an F-linear combination of elements of the form $e_{(k)} g_{\omega_3} g_{v_3}$ where $v_3 \in \mathfrak{B}_k$ and $\omega_3 \in \Sigma_{2k+1,n}$. This means

$$e_{(k)} g_{v_1} g_{u_2} g_{\omega_2} e_{(l)} g_{v_2} = \sum_{\substack{\omega_3 \in \Sigma_{2k+1,n} \\ v_3 \in \mathfrak{B}_k}} a_{(\omega_3, v_3)} e_{(k)} g_{\omega_3} g_{v_3} + a, \tag{3.8}$$

where $a_{(\omega_3,v_3)}$ are coefficients in F and a is an F-linear combination in J_{k+1}.

Now, the product of g_{h_1} and g_{h_2} is computed as follows.

$$\begin{aligned} g_{h_1}\cdot g_{h_2} &= (g_{u_1}e_{(k)}g_{\omega_1}g_{v_1})\cdot(g_{u_2}e_{(l)}g_{\omega_2}g_{v_2}) \\ &= g_{u_1}g_{\omega_1}(e_{(k)}g_{v_1}\cdot g_{u_2}g_{\omega_2}e_{(l)}g_{v_2}) \\ &\overset{(3.8)}{=} g_{u_1}g_{\omega_1}\Big(\sum_{\substack{\omega_3\in\Sigma_{2k+1,n}\\ v_3\in\mathfrak{B}_k}} a_{(\omega_3,v_3)}e_{(k)}g_{\omega_3}g_{v_3}+a\Big) \\ &= \sum_{\substack{\omega_3\in\Sigma_{2k+1,n}\\ v_3\in\mathfrak{B}_k}} a_{(\omega_3,v_3)}g_{u_1}g_{\omega_1}e_{(k)}g_{\omega_3}g_{v_3}+g_{u_1}g_{\omega_1}a, \end{aligned} \tag{3.9}$$

with $g_{u_1}g_{\omega_1}a\in J_{k+1}$. By Definition 3.2 we obtain

$$\begin{aligned} &(g_{u_1}\bar{e}_{(k)}\otimes\bar{e}_{(k)}g_{v_1}\otimes g_{\omega_1})\cdot(g_{u_2}\bar{e}_{(l)}\otimes\bar{e}_{(l)}g_{v_2}\otimes g_{\omega_2}) \\ &\quad= f_k\big(f_k^{-1}(g_{u_1}\bar{e}_{(k)}\otimes\bar{e}_{(k)}g_{v_1}\otimes g_{\omega_1})\cdot f_l^{-1}(g_{u_2}\bar{e}_{(l)}\otimes\bar{e}_{(l)}g_{v_2}\otimes g_{\omega_2})\ mod\ J_{k+1}\big) \\ &\quad= f_k(g_{h_1}\cdot g_{h_2}\ mod\ J_{k+1}) \\ &\quad\overset{(3.9)}{=} f_k\Big(\sum_{\substack{\omega_3\in\Sigma_{2k+1,n}\\ v_3\in\mathfrak{B}_k}} a_{(\omega_3,v_3)}g_{u_1}g_{\omega_1}g_{\omega_3}e_{(k)}g_{v_3}\Big) \\ &\quad\overset{f_k}{=} \sum_{\substack{\omega_3\in\Sigma_{2k+1,n}\\ v_3\in\mathfrak{B}_k}} [N]^k(g_{u_1}\bar{e}_{(k)}\otimes\bar{e}_{(k)}g_{v_3}\otimes a_{(\omega_3,v_3)}g_{\omega_1}g_{\omega_3}) \\ &\quad= g_{u_1}\bar{e}_{(k)}\otimes b\otimes g_{\omega_1}c, \end{aligned}$$

where $b:=\sum_{\substack{\omega_3\in\Sigma_{2k+1,n}\\ v_3\in\mathfrak{B}_k}}\bar{e}_{(k)}g_{v_3}$ and $c:=\sum_{\substack{\omega_3\in\Sigma_{2k+1,n}\\ v_3\in\mathfrak{B}_k}}[N]^k a_{(\omega_3,v_3)}g_{\omega_3}$. □

If $k=l$ then Definition 3.2 recovers Definition (3.4).

Proof. Keep notations as in the last lemma. For $k=l$ we get

$$\begin{aligned} &(g_{u_1}\bar{e}_{(k)}\otimes\bar{e}_{(k)}g_{v_1}\otimes g_{\omega_1})\cdot(g_{u_2}\bar{e}_{(k)}\otimes\bar{e}_{(k)}g_{v_2}\otimes g_{\omega_2}) \\ &\quad= f_k\big(f_k^{-1}(g_{u_1}\bar{e}_{(k)}\otimes\bar{e}_{(k)}g_{v_1}\otimes g_{\omega_1})\cdot f_k^{-1}(g_{u_2}\bar{e}_{(l)}\otimes\bar{e}_{(k)}g_{v_2}\otimes g_{\omega_2})\ mod\ J_{k+1}\big) \\ &\quad= f_k(g_{h_1}\cdot g_{h_2}\ mod\ J_{k+1}). \end{aligned} \tag{3.10}$$

For $k=l$, the definition of the bilinear form ϕ yields

$$\phi(\bar{e}_{(k)}g_{v_1}\otimes g_{u_2}\bar{e}_{(k)})\bar{e}_{(k)}\overset{\phi}{=}[N]^{2k}e_{(k)}g_{v_1}g_{u_2}e_{(k)}\quad mod\ J_{k+1}. \tag{3.11}$$

Hence,

$$\begin{aligned} g_{h_1}\cdot g_{h_2}\ mod\ J_{k+1} &\equiv (g_{u_1}e_{(k)}g_{\omega_1}g_{v_1})\cdot(g_{u_2}e_{(k)})g_{\omega_2}g_{v_2}\ mod\ J_{k+1} \\ &\equiv g_{u_1}g_{\omega_1}(e_{(k)}g_{v_1}\cdot g_{u_2}e_{(k)}g_{\omega_2}g_{v_2})\ modJ_{k+1} \\ &\overset{(3.11)}{=} g_{u_1}g_{\omega_1}([N]^{-2k}\phi(\bar{e}_{(k)}g_{v_1}\otimes g_{u_2}\bar{e}_{(k)})\bar{e}_{(k)})g_{\omega_2}g_{v_2}. \end{aligned} \tag{3.12}$$

Now substituting (3.12) into Formula (3.10) yields

$$
\begin{aligned}
&(g_{u_1}\bar{e}_{(k)} \otimes \bar{e}_{(k)}g_{v_1} \otimes g_{\omega_1}) \cdot (g_{u_2}\bar{e}_{(k)} \otimes \bar{e}_{(k)}g_{v_2} \otimes g_{\omega_2}) = \\
&\qquad = f_k\big(g_{u_1}g_{\omega_1}([N]^{-2k}\phi(\bar{e}_{(k)}g_{v_1} \otimes g_{u_2}\bar{e}_{(k)})\bar{e}_{(k)})g_{\omega_2}g_{v_2}\big) \\
&\qquad \overset{f_k}{=} [N]^{-k}(g_{u_1}\bar{e}_{(k)} \otimes \bar{e}_{(k)}g_{v_2}) \otimes g_{\omega_1}\phi(\bar{e}_{(k)}g_{v_1} \otimes g_{u_2}\bar{e}_{(k)})g_{\omega_2}.
\end{aligned}
$$

□

Altogether we have proved the following theorem.

Theorem 3.6. *The q-Brauer algebra $Br_n(N)$ is an iterated inflation of the Hecke algebras of symmetric groups. More precisely: as a free F- module, $Br_n(N)$ is equal to*

$$H_n(q) \oplus (U_1^* \otimes_F U_1 \otimes_F H_{3,n}(q)) \oplus (U_2^* \otimes_F U_2 \otimes_F H_{5,n}(q)) \oplus \dots,$$

and the iterated inflation starts with $H_n(q)$, inflates it along $U_1^* \otimes_F U_1 \otimes_F H_{3,n}(q)$ and so on, and ends with an inflation of $F = H_{n+1,n}(q)$ or $F = H_{n,n}(q)$ as bottom layer, depending on whether n is even or odd.

Suppose that Λ is a commutative noetherian ring which contains R as a subring with the same identity. If q, r and $[N]$ are invertible in Λ, then the q-Brauer algebra $Br_n(N)$ over the ring Λ is cellular with respect to the involution i. The proof of this Corollary comes from Lemma 2.1 and Theorem 3.6.

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TÓM TẮT

MỘT DÃY NÂNG LẶP CHO CÁC ĐẠI SỐ Q-BRAUER

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Trong bài báo này, áp dụng kĩ thuật của Koenig và Xi trong tài liệu [5] chúng tôi xây dựng một dãy nâng lặp cho các đại số q-Brauer. Dãy nâng lặp này sau đó được sử dụng để chỉ ra rằng, đại số q-Brauer có cấu trúc Cellula.
Từ khóa: Đại số q-Brauer; Đại số có dãy nâng lặp; Đại số Cellula.